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Two new reductions methods for polynomial differential equations and applications to nonlinear PDEs

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HIGHLIGHTS

- Two new methods to reduce polynomial equations to first-order equations are provided.
- The procedure is applied to find new exact solutions for nonlinear PDEs.
- Computer algebra codes to obtain the reduced equations are included.

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ABSTRACT

For ordinary differential equations of the form $P(y, v, v', \dots, v^{(n)}) = 0$ which are polynomial in the variables $v, v', \dots, v^{(n)}$ two new reduction methods to first-order equations are considered. The reduced equations are of the forms $v' = Q_1(y, v^{1/q})$ and $v' = \left(Q_2(y, v)\right)^{1/q}$, where Q_1 and Q_2 are two polynomials of degree p in $v^{1/q}$ and v, respectively, whose coefficients depend on y. In contrast to most of the known reduction methods of these types, which use either $q = 1$ or $q = 2$, in this paper the values of the positive integers p, q are not predetermined. A procedure to obtain the possible values of the integers for which a reduction of any of these types may exist is provided. As a consequence, new reductions that cannot be obtained by other known methods may be found.

The new methods have been applied to obtain some reductions and, consequently, new solutions for three polynomial ordinary differential equations related to well-known equations in mathematical physics: the Kuramoto–Sivashinsky equation, a generalized Benney equation and a 5th-order KdV equation. Some pieces of computer algebra code, written in Maple and implementing the underlying algorithms to derive the reductions, are also included.

1. Introduction

In the mathematical literature, there are a large number of papers devoted to reductions of Partial Differential Equations (PDEs): i.e. simpler equations whose solutions directly provide solutions for the given PDE. Many of these reductions correspond to Ordinary Differential Equations (ODEs) of the form

$$P(y, v, v', \dots, v^{(n)}) = 0, \tag{1}$$

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where $P(y, v, v' \ldots, v^{(n)})$ is a polynomial in the variables $v, v', \ldots, v^{(n)}$ whose coefficients are functions of the independent variable y. Very frequently, these polynomial equations are still very difficult to solve and further reductions become necessary.

There are several general methods that could be used for obtaining these further reductions of (1). The most commonly used approach is the theory of Lie point symmetries [1]. However, since there are equations that lack Lie point symmetries, some additional reduction processes have been recently developed from the existence of either nonlocal symmetries [2] or $\lambda-$symmetries [3]. Although these methods are available for any ODE, the corresponding determining equations may be very difficult to solve.

For obtaining reductions to first-order ODEs, in the last years some different classes of direct reduction schemes have been developed in the literature. For most of them the objective has been to determine first-order ODEs of any of the predetermined forms

$$v' = \sum_{i=0}^{k} a_i v^i, \tag{2}$$

$$(v')^2 = \sum_{i=0}^{k} a_i v^i, \tag{3}$$

where k is an *a priori* fixed positive integer ($k = 2$ or $k = 4$ in most cases) and the coefficients a_i, $0 \le i \le k$, are constant, to be determined by the corresponding method. Some of the known methods are: the *tanh* or the *sech method* [4], the *extended mapping* method [5], the Jacobi elliptic method [6], the *simplest equation method* [7].

These procedures of reduction have been recently generalized (see [8]) to obtain reductions of any of the form (2) or (3) where the coefficients a_i, $0 \le i \le k$, are functions of the independent variable y.

In this paper we consider two new classes of reductions which are strictly more general than the first-order reductions we have mentioned and, as far as we know, they have not been considered previously. For any pair p, q of positive integers it is investigated the existence of first-order reductions of any of the forms

$$v' = \sum_{i=0}^{p} a_i(y) v^{\frac{i}{q}}, \tag{4}$$

$$(v')^q = \sum_{i=0}^{p} a_i(y) v^i, \tag{5}$$

where p and q are arbitrary positive integers. When the a_i are constant, it is clear that for $q = 1$ the reductions (4), (5) and (2) are the same and for $q = 2$ the reduction (5) is the same as (3).

Obviously, it is not feasible to check if for every pair p, q of positive integers there is a reduction of the form (4) or (5) for (1). One of the main difficulties in dealing with these reductions is to determine the values of p, q for which a reduction may exist. This is investigated in Section 3 but our discussion requires, firstly, to know how the methods work in practice. Although the implementation of the methods requires given values of p, q, the basic ideas of the methods are independent of these values and they are provided in Section 2. This section also shows how the coefficients in (4) or (5) may be computed through the solutions of a set of determining equations.

Once it is shown how the methods work in practice, in Section 3 we include a procedure to determine the prospective values of p, q for which a non-trivial reduction of the form (4) or (5) may, *a priori*, exist. This analysis strongly reduces the number of possible cases to be considered. Even though the determination of prospective values of p, q for reductions of the form (4) differs from that for reductions of the form (5), a remarkable fact is that both reductions admit the same prospective pairs.

For each prospective pair (p, q), it must be investigated if these values provide equations of the form (4) or (5) whose solutions are also solutions of (1). This may be a very arduous task and the use of symbolic programs (such as Maple, Mathematica, etc.) becomes completely necessary. A model of computer code, written in Maple, for the two types of reductions we have considered is included and explained in Appendix.

A detailed account of the two mentioned methods is performed in Section 4 for the ODE that corresponds to the travelling-wave solutions of the generalized Kuramoto–Sivashinsky equation. Similar studies have been done for ODEs derived from the Benney equation (Section 5) and a 5th-order KdV equation (Section 6).

Some of the reductions we have found in this paper cannot be obtained by means of previous reduction methods. In fact, we have obtained reductions with $q = 3$ for the three considered PDEs, $q = 4$ for the 5th-order KdV equation and $q = 7$ for the Benney equation.

In Appendix we describe the whole procedure for obtaining the reductions of this paper, by using Maple. Although the presented code corresponds to the ODE that satisfies the travelling-wave solutions of the generalized Kuramoto–Sivashinsky equation, the necessary code for the remaining equations can be obtained with minor changes in the initial code.