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On the convergence rate of Clenshaw–Curtis quadrature for integrals with algebraic endpoint singularities

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ABSTRACT

In this paper, we are concerned with Clenshaw–Curtis quadrature for integrals with algebraic endpoint singularities. An asymptotic error expansion and convergence rate are derived by combining a delicate analysis of the Chebyshev coefficients of functions with algebraic endpoint singularities and the aliasing formula of Chebyshev polynomials. Numerical examples are provided to confirm our analysis.

1. Introduction

The evaluation of the definite integral

$$I[f] := \int_{-1}^{1} f(x)dx, \tag{1.1}$$

is one of the fundamental and important research topics in the field of numerical analysis [1–3]. Given a set of distinct nodes $\{x_k\}_{k=0}^{n}$, an interpolatory quadrature rule of the form

$$Q_n[f] := \sum_{k=0}^{n} w_k f(x_k), \tag{1.2}$$

can be constructed to approximate the above integral by requiring $I[f] = Q_n[f]$ whenever $f(x)$ is a polynomial of degree n or less. To achieve fast convergence, quadrature nodes with the Chebyshev density $\mu(x) = (1-x^2)^{-1/2}$ are preferable and ideal candidates are the roots or extrema of classical orthogonal polynomials such as Chebyshev and Legendre polynomials.

Clenshaw–Curtis quadrature is an interpolatory quadrature formula based on the Chebyshev points, i.e.,

$$x_k = \cos(k\pi/n), \qquad k = 0, \ldots, n. \tag{1.3}$$

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It has several remarkable features. First, it is exact for polynomials of degree up to n, however, its performance for differentiable functions is comparable with the $(n+1)$-point Gauss–Legendre quadrature which is exact for polynomials of degree up to $2n+1$ (see, e.g., [4–7]). Second, fast algorithms are available for its quadrature weights and therefore the implementation of the formula is quite robust (see [8,9]). Third, its quadrature nodes (1.3) are nested which makes it a suitable method in an adaptive environment. Due to these attractive properties, Clenshaw–Curtis quadrature is quite competitive with its Gauss–Legendre counterpart and is widely used for numerical integration (see [10,11] for more discussions).

Integrals with singularities at one or both endpoints, e.g., the integrand $f(x)$ has endpoint singularities at $x = \pm 1$, arise in diverse scientific applications. During the past few decades, convergence rate and asymptotic error expansion of Gauss–Legendre quadrature for such integrals have received considerable attention (see, e.g., [12–18]). For example, for the integrand $f(x) = (1-x)^{\alpha} g(x)$ where $\mathrm{Re}(\alpha) > -1$ is not an integer and $g(x)$ is analytic inside a neighborhood of the interval $[-1, 1]$, it was shown that the convergence rate of an n-point Gauss–Legendre quadrature is $\mathcal{O}(n^{-2\alpha-2})$. Moreover, a refined result on the asymptotic error expansion was given in [18]

$$E_n^{GL}[f] \sim \sum_{k=1}^{\infty} a_k h^{\alpha+k}, \quad h = \left(n+\frac{1}{2}\right)^{-2}, \tag{1.4}$$

where $E_n^{GL}[f]$ denotes the error of the n-point Gauss–Legendre quadrature.

While the question on the convergence rate of Gauss–Legendre quadrature when applied to integrals with endpoint singularities has been well answered, the same question for Clenshaw–Curtis quadrature has been lingering for quite a while and still remains unanswered. Trefethen mentioned this in the introduction of his well-known myth-dispelling paper [6, p. 68]. But singularity is not the focus of that paper. In this paper, we shall try to answer this question and restrict our attention to integrals that have algebraic endpoint singularities. By carefully exploiting the asymptotic character of Chebyshev coefficients of functions with algebraic endpoint singularities, we derive an asymptotic error expansion and the convergence rate of Clenshaw–Curtis quadrature (see Theorem 3.1). It is found that Clenshaw–Curtis quadrature converges at a comparable rate as its Gauss–Legendre counterpart.

We emphasis that the purpose of this work is to sharpen a theoretical understanding of Clenshaw–Curtis quadrature for integrals with endpoint singularities and it is not the aim to study the evaluation of singular integrals. In fact, from a computational point of view, alternative methods such as generalized Gaussian quadrature or quadrature formulas obtained by variable transformations are more preferable (see, e.g., [19–21]).

The rest of the paper is organized as follows. In Section 2, we present a delicate analysis on the asymptotic expansion of Chebyshev coefficients of functions with algebraic endpoints singularities. We derive an asymptotic error expansion of Clenshaw–Curtis quadrature in Section 3 and give the detailed proof in Appendix.

2. Asymptotic expansion of Chebyshev coefficients of functions with algebraic endpoints singularities

In this section, we deduce our first main result, an explicit asymptotic formula for the Chebyshev coefficients of functions with one or two algebraic endpoint singularities. For this purpose, we start with a useful lemma.

Lemma 2.1. *Assume that $\gamma, \delta > -1$, $h(x)$ is real and $h(x) \in C^{\infty}[a, b]$. Then, for large λ,*

$$\int_a^b (x-a)^{\gamma}(b-x)^{\delta} h(x) e^{i\lambda x} dx \sim e^{i\lambda a} \sum_{k=0}^{\infty} \frac{\psi^{(k)}(a) e^{i\frac{\pi}{2}(k+\gamma+1)} \Gamma(k+\gamma+1)}{\lambda^{k+\gamma+1} k!} - e^{i\lambda b} \sum_{k=0}^{\infty} \frac{\phi^{(k)}(b) e^{i\frac{\pi}{2}(k-\delta+1)} \Gamma(k+\delta+1)}{\lambda^{k+\delta+1} k!}, \tag{2.1}$$

where $\psi(x) = (b-x)^{\delta} h(x)$ and $\phi(x) = (x-a)^{\gamma} h(x)$.

Proof. The first proof of this result was given in [22]. The idea was based on the neutralizer functions together with integration by parts. If $h(x)$ is analytic in a neighborhood of the interval $[a, b]$, an alternative proof based on the contour integration was given in [23]. □

Let $T_n(x)$ denote the Chebyshev polynomial of the first kind of degree n, defined by

$$T_n(\cos\theta) = \cos(n\theta), \quad n \geq 0.$$

If a function $f(x)$ satisfies a Dini–Lipschitz condition on the interval $[-1, 1]$ then it can be expanded uniformly as [24, Theorem 5.7]

$$f(x) = \sum_{n=0}^{\infty}{}' a_n T_n(x), \quad a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{f(x) T_n(x)}{\sqrt{1-x^2}} dx, \tag{2.2}$$