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A numerical scheme for a singular control problem: Investment–consumption under proportional transaction costs

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ABSTRACT

This paper concerns the numerical solution of a fully nonlinear parabolic double obstacle problem arising from a finite portfolio selection with proportional transaction costs. We consider optimal allocation of wealth among multiple stocks and a bank account in order to maximize the finite horizon discounted utility of consumption. The problem is mainly governed by a time-dependent Hamilton–Jacobi–Bellman equation with gradient constraints. We propose a numerical method which is composed of Monte Carlo simulation to take advantage of the high-dimensional properties and finite difference method to approximate the gradients of the value function. Numerical results illustrate behaviors of the optimal trading strategies and also satisfy all qualitative properties proved in Dai et al. (2009) and Chen and Dai (2013).

1. Introduction

This paper presents the numerical solution of an optimal investment–consumption problem in the presence of proportional transaction costs during a finite time period. Given a known initial wealth, the objective of an investor is to decide the best consumption and investment strategy which maximizes the expected discounted utility of consumption over the finite investment period. In the absence of transaction costs and for specific utility functions, the solution can be exactly obtained and an investor's optimal trading strategy is to maintain a constant proportion of wealth invested in risky stocks, which is called the *Merton proportion* shown by Merton [1]. This constant proportion depends on the investor's risk preference and also the market parameters. Merton's strategy, simply stated, is to continuously rebalance portfolio holdings in order to keep the fraction of investment in risky assets constant. However, in the presence of transaction costs, a continuous portfolio rebalancing process may incur infinite costs. Thus, the question arises: *what is the optimal strategy if there are transaction costs in the market?*

Transaction cost appears in different ways, as a fixed commission or a proportion to the size of trade. This paper deals with the case where there is only proportional transaction costs; for a review of constant cost or a mixture of both, see [2] and references therein. Magill and Constantinides [3] are the first to introduce proportional transaction costs into Merton's model. They provide a valuable insight on the optimal strategy; i.e. an investor should maintain the fraction of wealth in risky assets inside a so-called *no-trading region* and trading only takes place along the boundary of the no-trading region. As a consequence, the crucial question is: *how to identify the optimal no-trading region which corresponds to the optimal trading strategy?*

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Under certain restricted settings, this question has been partially answered. When the market is confined to consist of a single risky asset and a bank account, Davis and Norman [4] give a rigorous analysis of the classical Merton's problem with proportional transaction costs over infinite time horizon. The optimal policy is formulated as a nonlinear free boundary problem which separates the buying and the selling regions from the no-trading one. Their paper contains detailed characterization, both theoretical and numerical, of the value function and optimal policies under certain assumptions. Shreve and Soner [5] relax assumptions of Davis and Norman [4]'s problem, and apply the viscosity solution approach to provide regularity and existence results. Many other papers have carried out an asymptotic analysis including [6,7], and [8]. A thorough convergence proof for general utility functions is studied by Soner and Touzi [9], and an extension to several risky assets is considered by Possamaï et al. [10]. Other numerical schemes have been proposed by Tourin and Zariphopoulou [11,12] for general utility functions, and by Muthuraman and Kumar [13] for a model with more than one risky asset. Nevertheless, these papers only deal with the infinite horizon scenario where the no-trading region does not evolve in time, and are based on finite difference/element method which are not efficient in higher dimensions.

Theoretical analysis on the finite-time problem has been studied recently and is restricted to the no consumption case with a single risky asset. Liu [14] first shows analytical properties of the optimal investment problem with a deterministic finite horizon. Dai and Yi [15] establish a link between the singular control problem and the obstacle problem, and completely characterize the behaviors of the resulting free boundaries. Numerical solution of this optimal investment problem is proposed by Arregui and Vázquez [16]. More recently, there is a plethora of literature devoted to the characterization of optimal investment–consumption strategy. Dai et al. [17] consider the investment and consumption optimization decision in finite time horizon, and characterize the behaviors of free boundaries for a single risky asset case. Dai and Zhong [18] propose the penalty method to demonstrate the numerical solution to a singular control problem arising from portfolio selection with proportional transaction costs. Bichuch [19] provides a proof to the same problem with power utility function by expanding the value function into a power series, and obtains a "nearly optimal" strategy.

In the present paper, we propose a numerical scheme based on Monte Carlo simulation for the optimal investment–consumption problem with proportional transaction costs and deterministic time horizon. As discussed in the next section, the value function of such control problem is characterized by a *Hamilton–Jacobi–Bellman (HJB) equation*. The existing numerical schemes for this HJB equation in the literature including [11,12] and [13] are based on finite difference/element method, which are only practical in low dimensional problems. Moreover, the dimension can be higher in many applications, especially in finance problems. Thus, we propose a numerical technique that combines Monte Carlo simulation with finite difference discretization so as to solve the nonlinear double obstacle problem, and aim to characterize the free boundaries and qualitative properties of the solution.

Our numerical scheme is strongly motivated by the aforementioned work of Fahim et al. [20] who introduce the backward probabilistic numerical scheme combined with Monte Carlo and finite difference method for high-dimensional fully nonlinear partial differential equations. They decompose the scheme into two steps. First, the Monte Carlo step includes isolating the linear generator of some underlying diffusion process to split the PDE into this linear part and a remaining nonlinear one. Then, a projection method is employed to evaluate the remaining nonlinear part of the PDE. In this paper, we will modify the numerical method to incorporate the free boundaries on the no-trading region. Moreover, we will show that the proposed method can work in the case of correlated stocks. It is worth noticing that the type of free boundaries in this current problem is different from the obstacle problem such as the one in [21] and therefore the scheme developed in this paper is not in the same nature of Monte Carlo scheme. We believe the motivation behind this proposed method can be extended to various HJB for singular control problems.

This paper is organized as follows. In Section 2 the optimal investment and consumption problem with proportional transaction costs is presented. Section 3 is dedicated to some simplifications of the control problem in Section 2. The numerical scheme composed of Monte Carlo simulation and finite difference discretization is proposed in Section 4. In Section 5, we show that the implementation of the proposed numerical scheme is compatible with the theoretical results in [17] and [22] in a single risky asset or two risky assets cases. Several examples that illustrate performances of the proposed numerical method are also presented in this section. And Section 6 draws some conclusion.

2. The optimal investment–consumption problem

We consider an optimal investment–consumption problem in finite time horizon $T \in (0, \infty)$ with proportional transaction costs, the model being the same in [18] and [22].

Suppose a continuous time market consisting of one risk-free asset and multiple risky assets available for investment. The risk-free asset (bank account), denoted by S_t^0, pays an interest rate $r > 0$ continuously and thus can be expressed as

$$dS_t^0 = rS_t^0 dt. \tag{2.1}$$

Let N be the number of available risky investments, called "stocks" hereafter. The N stocks have constant mean rates of return $\alpha_1, \alpha_2, \ldots, \alpha_N$. We denote the vector of N stock prices by $S_t = (S_t^1, S_t^2, \ldots, S_t^N)'$ and the mean rates of return by $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)'$. The evolution of stocks can be written as

$$dS_t = \mathbf{diag}(S_t)(\alpha dt + \sigma dB_t), \tag{2.2}$$