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SUPERCONVERGENCE OF THE LOCAL DISCONTINUOUS GALERKIN METHOD FOR THE SINE-GORDON EQUATION IN ONE SPACE DIMENSION

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Abstract. In this paper, we present superconvergence results for the local discontinuous Galerkin (LDG) method for the sine-Gordon nonlinear hyperbolic equation in one space dimension. We identify a special numerical flux and a suitable projection of the initial conditions for the LDG scheme for which the L^2-norm of the LDG solution and its spatial derivative are of order $p+1$, when piecewise polynomials of degree at most p are used. Our numerical experiments demonstrate optimal order of convergence. We further prove superconvergence toward particular projections of the exact solutions. More precisely, we prove that the LDG solution and its spatial derivative are $\mathcal{O}(h^{p+3/2})$ super close to particular projections of the exact solutions, while computational results show higher $\mathcal{O}(h^{p+2})$ convergence rate. Our analysis is valid for arbitrary regular meshes and for P^p polynomials with arbitrary $p \geq 1$. Numerical experiments validating these theoretical results are presented.
Keywords: Sine-Gordon equation; local discontinuous Galerkin method; superconvergence; projections; error estimates.

1. Introduction

In this paper, we present a superconvergent local discontinuous Galerkin (LDG) method for the one-dimensional sine-Gordon (SG) nonlinear hyperbolic equation

$$u_{tt} + \beta u_t + \sin(u) = u_{xx} + f(x,t), \quad x \in [a,b], \quad t > 0, \tag{1.1a}$$

subject to the initial conditions

$$u(x,0) = g(x), \quad u_t(x,0) = h(x), \quad x \in [a,b], \tag{1.1b}$$

where $u = u(x,t)$ represents the wave displacement at position x and time t. The parameter β is the so-called dissipative term, which is assumed to be a real number with $\beta \geq 0$. When $\beta = 0$, (1.1a) reduces to the undamped sine-Gordon equation in one space dimension, while when $\beta > 0$, to the damped one. Here $g(x)$ and $h(x)$ are the wave modes or kinks and velocity functions, respectively. In our analysis, the initial conditions are assumed to be sufficiently smooth functions on $[a,b]$. It is worthwhile to note that the SG equation (1.1a) is a particular case of the Klein-Gordon equation $u_{tt} + \beta u_t + V'(u) = u_{xx}$ in which the nonlinear force is given by $V'(u) = \sin(u)$. For the sake of simplicity, only periodic boundary conditions are considered throughout this work

$$u(a,t) = u(b,t), \quad u_x(a,t) = u_x(b,t), \quad t \geq 0. \tag{1.1c}$$

We would like to mention that this assumption is not essential since we do not use Fourier analysis. For instance, if other boundary conditions (Dirichlet or mixed boundary conditions) are chosen, the LDG method can be easily designed; see [4] for some discussion.

The nonlinear sine-Gordon equation arises in many different applications such as propagation of fluxion in Josephson junctions, differential geometry, stability of fluid motion, nonlinear physics, and applied sciences; see [35, 40, 39, 8, 26]. Other physical applications can be found in the review article by Barone *et al.* [8]. The term $\sin(u)$ is the Josephson current across an insulator between two superconductors [11].

In the literature several schemes have been developed for the numerical solution of sine-Gordon equation [36, 41, 42, 28, 29, 32, 23, 18, 24, 47, 9, 37, 43, 30, 1, 25, 31, 27, 10, 33, 7]. Among these methods, the finite-difference method, the finite-element method, the pseudospectral technique, the domain decomposition method. In this work, we solve the SG equation using the LDG method in space and the classical fourth-order explicit Runge-Kutta method in time. The LDG finite element method is an extension of the discontinuous Galerkin (DG) method aimed at solving ordinary and partial differential equations containing higher than first-order spatial derivatives. The DG method is a class of finite element methods using completely discontinuous piecewise polynomials for the numerical solution and the test functions. It was first introduced in 1973 by Reed and Hill [38], in the framework of neutron linear transport. DG method combines many attractive features of the classical finite element and finite volume methods. It is a powerful tool for approximating some differential equations which model problems in physics, especially in fluid dynamics or electrodynamics. With discontinuous finite element bases, they capture discontinuities in, *e.g.*, hyperbolic systems with high accuracy and efficiency; simplify adaptive $h-$, $p-$, $r-$, refinement and produce efficient parallel solution procedures. The LDG method for solving convection-diffusion problems was first introduced by Cockburn and Shu in [22]. Since then, DG methods have been successfully applied to solve various partial differential equations