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Numerical integration of oscillatory Airy integrals with singularities on an infinite interval☆

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ABSTRACT

This work is devoted to the quadrature rules and asymptotic expansions for two classes of highly oscillatory Airy integrals on an infinite interval. We first derive two important asymptotic expansions in inverse powers of the frequency ω. Then, based on structure characteristics of the two asymptotic expansions in inverse powers of the frequency ω, both the so-called Filon-type method and the more efficient Clenshaw–Curtis–Filon-type method are introduced and analyzed. The required moments in the former can be explicitly expressed by the Meijer G-functions. The latter can be implemented in $O(N \log N)$ operations, based on fast Fourier transform (FFT) and fast computation of the modified moments. Here, we can construct two useful recurrence relations for computing the required modified moments accurately, with the help of the Airy's equation and some properties of the Chebyshev polynomials. Particularly, we also provide their error analyses in inverse powers of the frequency ω. Furthermore, the presented error analysis shows the advantageous property that the accuracy improves greatly as ω increases. Numerical examples are provided to illustrate the efficiency and accuracy of the proposed methods.

1. Introduction

In this paper we are concerned with the numerical evaluation of singular oscillatory infinite integrals of the forms

$$I_1[f] = \int_0^{+\infty} x^\alpha f(x)Ai(-\omega x)dx, \tag{1.1}$$

$$I_2[f] = \int_0^{+\infty} x^\alpha \ln(x)f(x)Ai(-\omega x)dx, \tag{1.2}$$

where $\alpha \leq 0$, f is a sufficiently smooth function on $[0, +\infty)$, ω is a positive parameter, and $Ai(z)$ is an Airy function [1, p. 446]. Moreover, the two integrals (1.1) and (1.2) arise widely in many areas of science and engineering such as astronomy, electromagnetics, acoustics, scattering problems, physical optics, electrodynamics, computerized tomography, and applied mathematics [2–7]. In particular, it should be noted that the transforms (1.1) and (1.2) are infinite integrals with singularities of algebraic or logarithmic type, and oscillatory kernel functions, respectively. In most of the cases, such integrals cannot be calculated analytically, and then one has to resort to numerical methods. Traditionally one would have to resolve the oscillations by taking several sub-intervals for each period, resulting in a scheme whose complexity would

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grow linearly with the frequency of the oscillations. Therefore, it is very difficult to approximate accurately using standard methods, e.g., Gaussian quadrature rule. It is also noteworthy that the computation of the integrals (1.1) and (1.2) is less studied. This motivates us to develop accurate and efficient quadrature rules for computing these integrals.

Here, we would also like to mention some papers related to the integrals considered in this article. In recent years, there has been tremendous interest in developing numerical methods for the nonsingular or singular Bessel integrals on finite interval. Many numerical schemes were developed for computing generalized Bessel transforms $\int_a^b f(x)J_m(\omega g(x))dx$ without singularity (see [8–18]). Moreover, for the Bessel transform $\int_0^1 x^\alpha(1-x)^\beta f(x)J_m(\omega x)dx$, $\alpha, \beta > -1$, with singularities at the two endpoints, some researchers developed a few quadrature rules such as the *orthogonal expansion method* [19], the *Filon-type method* [20], and the *Clenshaw–Curtis–Filon-type method* [20,21]. Xu and Xiang in [22,23] proposed the *Clenshaw–Curtis–Filon-type method* for computing the oscillatory Airy integrals $\int_0^1 x^\alpha(1-x)^\beta f(x)Ai(-\omega x)dx$, $\alpha, \beta > -1$, with singularities at the two endpoints, and the highly oscillatory finite Hankel transform $\int_0^1 f(x)H_\nu^{(1)}(\omega x)dx$, respectively. For the oscillatory singular integral $\int_0^b x^\alpha(b-x)^\beta f(x)H_\nu^{(1)}(kx)e^{i\omega x}dx$, $\beta > -1$, $\alpha-|\nu| > -1$, with singularities at the two endpoints, He et al. [24] and Xu [25] developed a *special Gauss-type method* and a *Clenshaw–Curtis–Filon-type method*, respectively. The first author of this paper and Ling, Ma in [26,27] also presented the *Clenshaw–Curtis–Filon-type method* for computing a wide range of singular integrals with many different oscillatory kernel functions. On the other hand, we should mention several related articles for computing infinite oscillatory integrals. As early as in 1976, Blakemore et al. [28] reviewed several numerical methods for computing infinite range oscillatory integrals. However, those methods in [28] converge slowly, and have to use an extrapolation technique to accelerate convergence. Recently, Hascelik in [29] presented an *asymptotic Filon-type method* for calculating $\int_a^{+\infty} f(x)e^{i\omega g(x)}dx$. Based on the ideas of [29], Chen in [30,31] gave efficient numerical methods to evaluate $\int_a^{+\infty} f(x)J_m(\omega x)dx$. However, the algorithm (3.9) in [31] to calculate the generalized moments by transforming the Chebyshev polynomial $T_j(x)$ into power series of x^j is quite unstable for large j. In addition, it is worth pointing out that Siraj-ul-Islam et al. propose a few efficient numerical methods for computing several kinds of highly oscillatory integrals [32–34].

For convenience, we only consider the evaluation of (1.1) and (1.2) for the case $\lim_{x\to+\infty} f^{(k)}(x) = A_k$ (constant), $k = 0, 1, 2, \ldots$. Additionally, we transfer (1.1) and (1.2) into the following forms

$$I_1[f] = I_{11}[f] + I_{12}[f],$$
$$I_2[f] = I_{21}[f] + I_{22}[f],$$

where

$$I_{11}[f] = \int_0^1 x^\alpha f(x)Ai(-\omega x)dx, \tag{1.3}$$

$$I_{12}[f] = \int_1^{+\infty} x^\alpha f(x)Ai(-\omega x)dx, \tag{1.4}$$

$$I_{21}[f] = \int_0^1 x^\alpha \ln(x)f(x)Ai(-\omega x)dx, \tag{1.5}$$

$$I_{22}[f] = \int_1^{+\infty} x^\alpha \ln(x)f(x)Ai(-\omega x)dx. \tag{1.6}$$

Here, the two finite singular oscillatory integrals $I_{11}[f]$ and $I_{21}[f]$ on $[0, 1]$ in (1.3) and (1.5) can be accurately calculated by the efficient algorithms provided in [22,26], respectively. Consequently, our main goal in this paper is to introduce and analyze two efficient quadrature rules for the integrals (1.4) and (1.6). One is the so-called Filon-type method. The other is the more efficient Clenshaw–Curtis–Filon-type method. In order to develop the two methods, and then give their error analysis, we can derive two important asymptotic expansions in inverse powers of the frequency ω. In addition, we construct two useful recurrence relations for computing the required modified moments. Here, the required modified moments can be accurately computed by using the forward recursion as long as $\omega \geq 2j$. Moreover, for $\omega < 2j$ Oliver's algorithm [35] and Lozier's algorithm [36] with the starting values and end values are numerically stable. Moreover, the interpolation coefficient a_k in (4.29) can be accurately computed by using an efficient algorithm (see [37]), based on fast Fourier transform (FFT). Therefore, the proposed Clenshaw–Curtis–Filon-type method can be efficiently implemented in $O(N \log N)$ operations, which avoids solving an ill-conditioned linear system with $O(N^3)$ operations. Not only for the well-behaved $f(x)$, more importantly, but also for the ill-behaved $f(x)$, the advantage of the presented Clenshaw–Curtis–Filon-type method, becomes very apparent once we use a huge number of nodes, exploiting the power of fast Fourier transform (FFT).

The organization of this paper is as follows. In the next section, two key asymptotic expansions in inverse powers of ω are derived. Then, based on structure characteristics of the asymptotic expansions, we present a Filon-type method and provide its error analysis. Here, The required moments can be explicitly expressed by the Meijer G-functions. Section 3 gives a Clenshaw–Curtis–Filon-type method and its error analysis. Particularly, by constructing two important recurrence relations, the required modified moments in the Clenshaw–Curtis–Filon-type method can be accurately and efficiently computed. Moreover, in Sections 2–3, some numerical examples are used to show the accuracy and efficiency of these quadrature rules. All these presented methods share an advantageous property that the error decreases greatly as ω increases.