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A uniformly almost second order convergent numerical method for singularly perturbed delay differential equations

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ABSTRACT

The purpose of this paper is to present a uniform finite difference method for the numerical solution of a second order singularly perturbed delay differential equation. The problem is solved by using a hybrid difference scheme on a Shishkin-type mesh. The method is shown to be uniformly convergent with respect to the perturbation parameter. Numerical experiments illustrate in practice the result of convergence proved theoretically.

1. Introduction

Singularly perturbed differential equations are of practical interest in models of instantaneous phenomena in the biosciences and control theory. Singularly perturbed delay differential equations are identified by those problems in which the highest derivative term is multiplied by a small parameter and involving one or more delay term. This type problems occur where the future depends not only on the immediate present, but also on the past history. Such problems govern frequently in the mathematical modeling of various practical phenomena, the study of bistable devices [1], first exit time problems in the modeling of the determination of expected time for the generation of action potential in nerve cells by random synaptic inputs in dendrites [2], description of the human pupil–light reflex [3], a variety of models for physiological processes or diseases [4,5], evolutionary biology [5], variational problems in control theory [6].

Motivated by the works of [7,8], we consider the following singularly perturbed delay initial value problem: find $u \in C^0\left(\bar{I}\right) \cap C^1(I) \cap C^2(I^*)$ such that

$$Tu(t) \equiv \varepsilon u''(t) + a(t)u'(t) + f(t, u(t), u(t-r)) = 0, \quad t \in I^*, \tag{1.1}$$

$$u(t) = \varphi(t), \quad t \in I_0, \tag{1.2}$$

$$u'(0) = A/\varepsilon, \tag{1.3}$$

where $I = (0, T]$, $I^* = \cup_{p=1}^{m} I_p$, $I_p = \left\{t : r_{p-1} < t < r_p\right\}$, $r_p = pr$ for $1 \le p \le m$ and $I_0 = [-r, 0]$. $0 < \varepsilon \le 1$ is the perturbation parameter, $r > 0$ is a constant delay, A is a constant, $a(t) \ge \alpha > 0$, $\varphi(t)$ and $f(t, u, v)$ are given sufficiently smooth functions satisfying certain regularity conditions in $\bar{I}$ and $\bar{I} \times \mathbb{R}^2$, respectively, and moreover

$$\left|\frac{\partial f}{\partial u}\right| \le M_1 \text{ and } \left|\frac{\partial f}{\partial v}\right| \le M_2, \tag{1.4}$$

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where M_1 and M_2 are positive constants. Under these assumptions, this type of problems (1.1)–(1.3) has a unique solution on the interval I (see [7,8]).

The numerical study of first-order singularly perturbed delay differential equation can be found in [9–14] and references therein. The numerical study of second-order singularly perturbed delay differential equations is initiated in [7,8,15,16]. But they have focused on first order uniformly convergent schemes. In [17], authors have considered a second order uniformly convergent scheme for a singularly perturbed initial value problem without delay. Due to the delay term, the boundary and interior layers may occur in the exact solution. Hence the singularly perturbed delay differential problem (1.1)–(1.3) is different from the singularly perturbed problem considered in [17].

In this article, our main objective is to construct a higher order hybrid numerical scheme for the singularly perturbed delay differential problem (1.1)–(1.3). Our scheme combines second-order difference schemes on the fine mesh with the midpoint upwind scheme on the coarse mesh, which is a modification of the difference scheme used in [18–20].

The rest of the paper is organized as follows. In Section 2, we state some important properties of the exact solution. In Section 3, we describe the hybrid finite difference scheme and introduce a Shishkin-type mesh. In Section 4 we analyze the convergence properties of the scheme. Numerical examples are presented in Section 5.

Notation. Throughout this paper, we assume C denotes a generic positive constant that may take different values in the different formulas, but is always independent of N and the mesh.

2. The continuous problem

Here we show some properties of the solution of (1.1)–(1.3), which are needed in later sections for the analysis of the appropriate numerical solution. For any continuous function $g(t)$, we use $\|g\|_\infty$ for the continuous maximum norm on the corresponding interval.

In the following error analysis, we need the bounds of the fourth order derivative of the exact solution. Under the assumptions $\varphi \in C^3(I_0)$, $a \in C^3(\bar{I}_1)$ and $f \in C^3(\bar{I}_1 \times R^2)$, we have at least $u \in C^4(\bar{I}_1)$ by using the Peano Theorem. Recursively, under the conditions $u \in C^4(\bar{I}_p)$, $a \in C^3(\bar{I}_{p+1})$ and $f \in C^3(\bar{I}_{p+1} \times R^2)$ with $1 \le p < m$, we also have at least $u \in C^4(\bar{I}_{p+1})$. Hence,

$$\varphi \in C^3(I_0), \quad a \in C^3(\bar{I}) \text{ and } f \in C^3(\bar{I} \times R^2) \tag{2.1}$$

imply $u \in C^4(\bar{I})$ at least.

Lemma 1. *Under the assumptions (2.1), the solution u of problem (1.1)–(1.3) satisfies the following bounds*

$$\left|u^{(k)}(t)\right| \le C\left\{1 + \varepsilon^{-k} e^{-\alpha t/\varepsilon}\right\}, \qquad t \in \bar{I}_1, \quad 0 \le k \le 4, \tag{2.2}$$

$$\left|u^{(k)}(t)\right| \le C\left\{1 + \varepsilon^{1-k} e^{-\alpha(t-r_{p-1})/\varepsilon}\right\}, \qquad t \in \bar{I}_p, \ 2 \le p \le m, \ 0 \le k \le 4. \tag{2.3}$$

Proof. Our proof is given step by step. On the first interval $\bar{I}_1$, the results (2.2) are easy to obtain from [17, Lemma 1]. Since $u \in C^1(I)$, from (2.2) we have

$$\left|u'(r_1)\right| \le C\left(1 + \varepsilon^{-1} e^{-\alpha r_1/\varepsilon}\right) \le C\left(1 + \frac{1}{\alpha r_1} e^{-\alpha r_1/(2\varepsilon)}\right) \le C, \tag{2.4}$$

where we have used the following inequality

$$x^k e^{-x} \le C e^{-x/2} \qquad \text{for } x \ge 0 \text{ and } k \in \mathbb{R}^+.$$

Applying the same method as that in [17, Lemma 1] we can get

$$\left|u^{(k)}(t)\right| \le C\left\{1 + \varepsilon^{1-k} e^{-\alpha(t-r_1)/\varepsilon}\right\}, \qquad t \in \bar{I}_2, \ 0 \le k \le 4. \tag{2.5}$$

By using the same method we also can obtain the desired results (2.3) for $t \in \bar{I}_p$ with $3 \le p \le m$.

In order to prove that the numerical method is ε-uniform, more precise information on the behavior of the exact solution of problem (1.1)–(1.3) is needed. This is obtained by writing the solution in the form

$$u(t) = v(t) + w(t), \qquad t \in I \cup I_0, \tag{2.6}$$

where $v(t)$ and $w(t)$, respectively, are the regular and singular components of $u(t)$. The regular component $v(t)$ is the solution of

$$Tv(t) = 0, \qquad t \in I^* \tag{2.7}$$

with initial value conditions

$$v(t) = v_0(t) + \varepsilon v_1(t) + \varepsilon^2 v_2(t) + \varepsilon^3 v_3(t), \qquad t \in I_0, \tag{2.8}$$

$$v'(0) = v_0'(0) + \varepsilon v_1'(0) + \varepsilon^2 v_2'(0) + \varepsilon^3 v_3'(0), \tag{2.9}$$