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A fully discretised filtered polynomial approximation on spherical shells

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ABSTRACT

A fully implementable filtered polynomial approximation on spherical shells is considered. The method proposed is a quadrature-based version of a filtered polynomial approximation. The radial direction and the angular direction of the shells are treated separately with constructive filtered polynomial approximation. The approximation error with respect to the supremum norm is shown to decay algebraically for functions in suitable differentiability classes. Numerical experiments support the results.

1. Introduction

This paper is concerned with constructive global polynomial approximation on spherical shells. Problems on such domains naturally arise in a wide range of geosciences, and numerous computational methods are proposed [1–5]. Nonetheless, theoretical analysis does not seem to have attracted much attention. One recent result considered in [6] is a fully discretised polynomial approximation on spherical shells. The method considered there can be seen as an approximation of the L^2-orthogonal projection. However, L^2-projection is not the best choice when one wants a small point-wise error—recall how the Fourier series of f on the torus may fail to converge on any measure zero set if f is merely continuous [7]. This paper considers a method with good uniform convergence using *filtering*. Our ultimate goal is to construct a fully discretised filtered polynomial approximation method and analyse the errors.

A classical remedy for the failure of the Fourier series on the torus mentioned above is to use a smoothing (or *filtering*) process such as Cesáro sums, Lanczos smoothing, or the raised cosine smoothing [8]. An underlying idea is to smoothly truncate the series by multiplying the Fourier coefficients of higher order by a suitably small factor.

Filtered approximations have also been considered for other regions, including the sphere and the ball. A standard approach to show the convergence rate is to show the uniform boundedness of the family of linear operators defined by the approximation, which readily reduces the problem to the best polynomial approximation rate. To show such boundedness, the assumption that the Fourier coefficients are given exactly is often fully utilised. See [9–12] and references therein.

In most realistic applications the Fourier coefficients are not known, as integrals are not computable exactly. As an alternative, quadrature-based approximations of these filtered methods have been considered for various settings, particularly on the sphere [13,14]. The present paper considers a quadrature-based filtered polynomial approximation for spherical shells $\mathbb{S}_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^3 \mid r_{\mathrm{in}} \le \|\boldsymbol{x}\|_2 \le r_{\mathrm{out}} = r_{\mathrm{in}} + \varepsilon\}$, where $r_{\mathrm{in}} \le 1 \le r_{\mathrm{out}}$ and $r_{\mathrm{out}} - r_{\mathrm{in}} = \varepsilon > 0$ as a domain. That is, given a function f, we smoothly truncate its Fourier series by a suitable 'filter' h, and approximate Fourier coefficients

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by quadrature rules. The motivation is to propose an implementable technique with a good point-wise convergence. Our results give, to the best of our knowledge, the first theoretical results on constructive filtered polynomial approximation on spherical shells. Our method requires only point values of the function we approximate, and thus can be implemented exactly in a real number model of computation.

We regard $\mathbb{S}_\varepsilon$ as a product of the interval $[r_{\text{in}}, r_{\text{out}}]$ in the radial direction and the unit sphere S^2 in the angular direction. The product setting is natural, since in practice functions on a spherical shell vary on different scales in the radial and angular directions. For example, the mantle can be seen as a set of spherical layers with different characteristics [15–17]. Some properties of the atmosphere, such as the ionisation rate [18, p. 151], electric field [18, p. 155], depend strongly on the altitude, and hence vary rapidly in the radial direction.

For a continuous function $f \in C(\mathbb{S}_\varepsilon)$ we consider the approximation taking the following form. Let $J_k^{(\alpha,\beta)}$ $(\alpha, \beta > -1)$ be the Jacobi polynomial of degree k mapped affinely to $[r_{\text{in}}, r_{\text{out}}]$ from $[-1, 1]$, and $Y_{\ell m}$ be the spherical harmonics of degree ℓ and order m. More detailed definitions are given later. Let $h : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a function with a compact support that is non-increasing in each variable. Then, the method we propose takes the form

$$V_{KL}f := \sum_{k,\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h\left(\frac{k}{K}, \frac{\ell}{N}\right) c_{k\ell m} J_k^{(\alpha,\beta)} Y_{\ell m}. \tag{1.1}$$

Note that this is actually a finite sum. Here, the coefficients $\{c_{k\ell m}\} \subset \mathbb{R}$ are quadrature approximations of Fourier coefficients,

$$c_{k\ell m} \approx \frac{1}{\gamma_k^2} \int_{\mathbb{S}_\varepsilon} f J_k^{(\alpha,\beta)} Y_{\ell m}.$$

The quadrature approximation, the measure used in the integral, and the normalising constant γ_k are defined later.

Following [14,19], we shall call $V_{KL}f$ filtered hyperinterpolation of f on $\mathbb{S}_\varepsilon$, if the quadrature is of suitably high polynomial precision.

Our main result Corollary 5.4 gives error convergence orders of the method we propose in terms of the supremum norm. Our strategy for the proof is similar to [14,19] in that we reduce the error estimate to suitable best polynomial approximations. What differs in our setting is that we have the radial direction as well. We treat the error in the radial and angular directions separately by introducing the filtered hyperinterpolation operator R_K in the radial direction and A_L in the angular direction. The error $f - V_{KL}f$ in terms of the supremum norm turns out to be bounded by the sum of the error bounds for each direction.

The outline of this paper is as follows. Section 2 introduces notations we need. In Sections 3 and 4, we introduce the filtered hyperinterpolation approximations in the radial direction and the angular direction. Section 5 develops the filtered hyperinterpolation on spherical shells and analyses the error. We give numerical results in Section 6, and Section 7 concludes the paper.

2. Preliminaries

We set up some notations and introduce the problem we consider.

With $r_{\text{in}} \in (0, 1]$ and $r_{\text{out}} \in [1, \infty)$ $(r_{\text{in}} \neq r_{\text{out}})$, we consider a spherical shell $\mathbb{S}_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^3 \mid r_{\text{in}} \leq \|\boldsymbol{x}\|_2 \leq r_{\text{out}}\}$. We assume $r_{\text{out}} - r_{\text{in}} = \varepsilon > 0$. We use the spherical coordinate system

$$\boldsymbol{x} = r\boldsymbol{\sigma} = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta) \quad (r \in [0, \infty), \theta \in [0, \pi],\ \varphi \in [0, 2\pi)),$$

where $r = \|\boldsymbol{x}\|_2$, $\boldsymbol{\sigma} = \frac{\boldsymbol{x}}{r}$, and for $\theta \in \{0, \pi\}$ we let $\varphi = 0$.

For $\boldsymbol{\sigma} \in S^2$, we often write a function $f(\theta, \varphi)$ on the unit sphere S^2 as $f(\boldsymbol{\sigma})$.

In the following, we introduce orthogonal polynomials on the interval and the sphere. Further, we introduce the approximation method we consider.

2.1. Orthogonal polynomials

Let $J_k^* = J_k^{*(\alpha,\beta)}$ be the Jacobi polynomial of degree k with the parameters $\alpha, \beta > -1$ on $[-1, 1]$. Define J_k $(k = 0, \ldots, K)$ by

$$J_k(r) = J_k^*\left(\frac{2r - (r_{\text{in}} + r_{\text{out}})}{r_{\text{out}} - r_{\text{in}}}\right), \qquad r \in [r_{\text{in}}, r_{\text{out}}].$$

Let $w^*(x) := (1 - x)^\alpha (1 + x)^\beta$ $(x \in (-1, 1))$ be the weight function associated with $\{J_k^*\} = \{J_k^{*(\alpha,\beta)}\}$, that is, with $\gamma_k = \left(\int_{-1}^{1} (J_k^*(x))^2 w^*(x) \mathrm{d}x\right)^{\frac{1}{2}}$ we have

$$\int_{-1}^{1} J_j^*(x) J_k^*(x) w^*(x) \mathrm{d}x = \delta_{jk} \gamma_k^2, \tag{2.1}$$