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Solving linear and quadratic random matrix differential equations using: A mean square approach. The non-autonomous case

M.-C. Casabán *, J.-C. Cortés, L. Jódar

Instituto Universitario de Matemática Multidisciplinar, Building 8G, access C, 2nd floor, Universitat Politècnica de València, Camino de Vera s/n, 46022 València, Spain

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ABSTRACT

This paper is aimed to extend, the non-autonomous case, the results recently given in the paper Casabán et al. (2016) for solving autonomous linear and quadratic random matrix differential equations. With this goal, important deterministic results like the Abel–Liouville–Jacobi's formula, are extended to the random scenario using the so-called L_p-random matrix calculus. In a first step, random time-dependent matrix linear differential equations are studied and, in a second step, random non-autonomous Riccati matrix differential equations are solved using the hamiltonian approach based on dealing with the extended underlying linear system. Illustrative numerical examples are also included.

1. Introduction

In the recent paper [1] linear and quadratic random autonomous differential equations were motivated and studied in the L_p-random sense. In that paper, all the coefficients were assumed to be random matrices rather than matrix stochastic processes, hence in [1] coefficients do not depend on time. Based on the well-known linear hamiltonian approach, (see [2,3] for excellent references about Riccati differential equations and the hamiltonian approach), the solution of the initial value problem for a general class of Riccati random quadratic matrix equations is obtained in terms of the blocks of the solution stochastic process of the underlying random linearized problem.

In this paper, we address the solution in the L_p-random sense of the non-autonomous Riccati matrix differential initial value problem (IVP)

$$W'(t) + W(t)A(t) + D(t)W(t) + W(t)B(t)W(t) - C(t) = 0, \quad W(0) = W_0, \tag{1}$$

where the variable coefficient matrices $A(t) \in L_p^{n\times n}(\Omega)$, $D(t) \in L_p^{m\times m}(\Omega)$, $B(t) \in L_p^{n\times m}(\Omega)$, $C(t) \in L_p^{m\times n}(\Omega)$, the initial condition $W_0 \in L_p^{m\times n}(\Omega)$ and the unknown $W(t) \in L_p^{m\times n}(\Omega)$ are matrix stochastic processes whose size are specified in the superindexes and defined in certain $L_p^{r\times s}(\Omega)$ spaces, that will be specified later. It is important to underline that in (1), the meaning of the derivative $W'(t)$ is understood in the pth mean sense, that is, a kind of strong random convergence that it will be introduced in Section 2. It is convenient to highlight that using the $L_p^{r\times s}(\Omega)$-random approach is not equivalent to deal with the averaged deterministic problem based on taking the expectations in every entry of the matrices that define the

* Corresponding author.
E-mail addresses: macabar@imm.upv.es (M.-C. Casabán), jccortes@imm.upv.es (J.-C. Cortés), ljodar@imm.upv.es (L. Jódar).

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differential equation (1). Even more, from a practical point of view, it is more realistic to consider the random approach rather than the deterministic since when modelling input data of the Riccati equation (1) are usually fixed after measurements, hence having errors. We point out that the content of this paper may be regarded as a continuation of [1,4,5]. Finally, we highlight some recent and interesting contributions dealing with scalar random Riccati-type differential equations by means of $\mathrm{L}_p(\Omega)$-random calculus or alternative techniques [6,7], for example.

The organization of this paper is as follows. Section 2 is devoted to extend some stochastic results presented in Section 2 of [1] and to introduce new ones as well. These new results are addressed to establish a random analogous of the Abel–Liouville–Jacobi's formula that will play a key role to deal with the non-autonomous random case. In Section 3 the random non-autonomous matrix linear problem is treated, including the bilateral case. In Section 4 the random non-autonomous Riccati matrix equation is solved based on the extended underlying linear problem, including a procedure for the numerical solution inspired in the results of [4] that were obtained for the non-autonomous deterministic counterpart. In Section 5 the theoretical results obtained throughout the paper are illustrated by means of several numerical examples. Finally, conclusions are drawn in Section 6.

2. New results on L_p-random matrix calculus

The aim of this section is to establish new results belonging the so called $\mathrm{L}_p(\Omega)$-random matrix calculus that will be required later for solving both non-autonomous linear systems (see Section 3) and non-autonomous nonlinear random Riccati-type matrix differential equations of the form (1) (see Section 4). This section can be viewed as continuation of the contents introduced in [1, Sec.2]. For the sake of consistency, hereinafter we will keep the same notation introduced in [1]. For ease of presentation, it is convenient to remember that given a complete probability space, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathrm{L}_p^{m\times n}(\Omega)$ denotes the set of all real random matrices $X = (x_{i,j})_{m\times n}$ such as $x_{i,j} : \Omega \longrightarrow \mathbb{R}$, $1 \le i \le m$, $1 \le j \le n$, are real random variables (r.v.'s) satisfying that

$$\left\| x_{i,j} \right\|_p = \left(\mathrm{E}\left[|x_{i,j}|^p \right] \right)^{1/p} < +\infty, \quad p \ge 1, \tag{2}$$

where $\mathrm{E}[\cdot]$ denotes the expectation operator. It can be proved that $(\mathrm{L}_p^{m\times n}(\Omega), \|\cdot\|_p)$, where

$$\|X\|_p = \sum_{i=1}^{m}\sum_{j=1}^{n} \left\| x_{i,j} \right\|_p, \quad \mathrm{E}\left[|x_{i,j}|^p \right] < +\infty, \tag{3}$$

is a Banach space. Notice that no confusion is possible between the common notation used for the $\|\cdot\|_p$ in (2) and in (3) because they act on scalar r.v.'s (denoted by lower case letters) and random matrices (denoted by capital case letters), respectively. In the case that $m = n = 1$, both norms are the same and $(\mathrm{L}_p^{1\times 1}(\Omega) \equiv \mathrm{L}_p(\Omega), \|\cdot\|_p)$ represents the Banach space of real r.v.'s with finite absolute moments of order p about the origin, being $p \ge 1$ fixed, [8]. In [9] a number of results corresponding to $p = 4$ (fourth random calculus) and its relationship with $p = 2$ (mean square calculus) are established and applied to solve scalar random differential equations. In [10] a scalar random Riccati differential equation whose nonlinear coefficient is assumed to be an analytic stochastic process is solved using the $\mathrm{L}_p(\Omega)$-random scalar calculus.

Given $\mathcal{T} \subset \mathbb{R}$, a family of t-indexed r.v.'s, say $\{x(t) : t \in \mathcal{T}\}$, is called a p-stochastic process (p-s.p.) if for each $t \in \mathcal{T}$, the r.v. $x(t) \in \mathrm{L}_p(\Omega)$. This definition can be extended to matrix s.p.'s $X(t) = (x_{i,j}(t))_{m\times n}$ of $\mathrm{L}_p^{m\times n}(\Omega)$, which are termed p-matrix s.p.'s, if $x_{i,j}(t) \in \mathrm{L}_p(\Omega)$ for every $1 \le i \le m$ and $1 \le j \le n$.

The definitions of continuity, differentiability and integrability of p-matrix s.p.'s follow in a straightforwardly manner using the $\|\cdot\|_p$-norm introduced in (3). As a simple but illustrative example that will be invoked later when showing more sophisticated examples in Section 5, we show how to prove the p-differentiability of a matrix s.p. of $\mathrm{L}_p^{n\times n}(\Omega)$.

Example 1. Let a be an absolutely continuous r.v. defined on the bounded interval (a_1, a_2), i.e., $a_1 \le a(\omega) \le a_2$ for every $\omega \in \Omega$, and let us denote by $f_a(a)$ the probability density function (p.d.f.) of the r.v. a. Let us define the following matrix s.p.

$$H(t;a) = \begin{bmatrix} h_{1,1}(t;a) & h_{1,2}(t;a) \\ h_{2,1}(t;a) & h_{2,2}(t;a) \end{bmatrix} = \begin{bmatrix} \exp(at) & \cosh(at) \\ \sinh(at) & \exp(-at) \end{bmatrix}, \quad t \in [0,T].$$

On the one hand, by the definition of the random matrix p-norm (see (3)) one gets

$$\|H(t;a)\|_p = \sum_{i=1}^{2}\sum_{j=1}^{2} \left\| h_{i,j}(t;a) \right\|_p = \|\exp(at)\|_p + \|\cosh(at)\|_p + \|\sinh(at)\|_p + \|\exp(-at)\|_p .$$

On the other hand, if we denote

$$M_{t,p} := \max\{M_{t,p}^{i,j} : 1 \le i,j \le 2\}, \quad \text{where } M_{t,p}^{i,j} := \max_{\omega\in\Omega}\{(h_{i,j}(t;a(\omega)))^p\} < +\infty.$$