ARTICLE IN PRESS
Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect
Journal of Computational and Applied Mathematics
journal homepage: www.elsevier.com/locate/cam

Geometrical definition of a continuous family of time transformations on the hyperbolic two-body problem

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ARTICLE INFO

Article history:
Received 28 November 2016
Received in revised form 11 March 2017

MSC:
70F05
70F10
70F15
70M20

Keywords:
Celestial mechanics
Orbital motion
Ordinary differential equations
Computational algebra

ABSTRACT

This paper is aimed to address the study of techniques focused on the use of a new set of anomalies based on geometric continuous transformations, depending on a parameter α, that includes the true anomaly. This family is an extension of the elliptic geometrical transformation to the hyperbolic case.

This transformation allows getting closed equations for the classical quantities of the hyperbolic two body problem both in the attractive and in the repulsive case.

In this paper, we obtain the link between hyperbolic functions of hyperbolic argument H to trigonometric functions for each temporal variable in the new family, including also the inverse relations. We also carry out the study, in the attractive case, of the minimization of the errors due to the choice of a temporal variable included in our family in the numerical integration by an appropriate choice of parameters. This study includes the analysis of the dependence on the parameter of integration errors in a great time span for several eccentricities as well as the study of local truncation errors along the region with true anomaly contained in the interval $[-\pi/2, \pi/2]$ around the primary for several values of the parameter.

1. Introduction

One of the most important topics in celestial mechanics is the study of the two-body problem. This problem includes the elliptic, parabolic and hyperbolic motions and its solution can be described through the orbital elements. For example, the third set of Brouwer and Clemence [1] $(a, e, i, \Omega, \omega, M)$ in the elliptic and hyperbolic case. The parabolic case is a borderline case when $e = 1$ where the major semiaxis becomes infinite so it can be replaced by the parameter p. The elliptic motion is the most important because it can be used to obtain a first approximation to the motion of the planets, natural and artificial satellites, periodic comets, etc.

The parabolic motion separates the regions of elliptical and hyperbolic motion. Parabolic motion is appropriate as a first approximation in the perihelion region to the orbital motion of the comets with eccentricity close to unity. In this case, from a few observations, we can determine a provisional parabolic orbit and then we can make a short-term tracking of the body to obtain more positions in order to improve the accuracy of its orbital elements, its eccentricity in particular, even if it is a periodic comet, because the method does not depend on the form of the orbit that is elliptical if it is periodic.

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http://dx.doi.org/10.1016/j.cam.2017.04.048
0377-0427/

Please cite this article in press as: J.A. López Ortí, et al., Geometrical definition of a continuous family of time transformations on the hyperbolic two-body problem, Journal of Computational and Applied Mathematics (2017), http://dx.doi.org/10.1016/j.cam.2017.04.048

While, in general, the hyperbolic motion is less important in celestial mechanics than the elliptical, it has a special interest in problems of astronautics where in many occasions the gravitational force of the planets can be used to move the spacecraft during a time in a hyperbolic orbit around them to direct the spacecraft to its target.

To study the motion of the spacecraft in hyperbolic motion it is appropriate the use of the numerical integrators. As in the elliptical case, the performance is good, although an adequate choice of the temporal variable can, in certain cases, increase the efficiency of these methods. In the hyperbolic case, the use of the natural time as integration variable presents a problem that becomes greater as the eccentricity approaches one. The problem is that at temporal regular intervals there is a much lower concentration of points in the region of the periapsis, in which velocity and curvature, are maxima than in remote regions where the motion is quasi-inertial.

The relative motion of the secondary with respect to the primary is defined by the second order differential equations:

$$\frac{d^2\vec{r}}{dt^2} = -\mu\frac{\vec{r}}{r^3} + \vec{F} \tag{1}$$

where $\vec{r}$ is the radius vector of the secondary, μ is the spaceflight constant given by $\mu = G(m_1 + m_2)$ in the gravitatory case where G is the gravitational constant, and m_1, m_2 the masses of primary and secondary, respectively, and $\vec{F}$ the perturbative force.

In the electrostatic case, we have $\mu = \left(K\frac{q_1}{m_1}\frac{q_2}{m_2} - G\right)(m_1+m_2)$ where q_1 and q_2 are the charges of primary and secondary body, M_1 and m_2 their masses and K is the Coulombian constant. In general, in this case, we can ignore the gravitational forces ($|G| \ll K$) and also the magnetic forces when $v^2/c^2 \ll 1$. In the case of the electrostatic forces, they are repulsive when the charges of two bodies have the same sign and attractive when they are opposite.

To integrate the system (1), it is necessary to know the initial values of the radius vector $\vec{r}_0$ and velocity $\vec{v}_0$.

The external branch of the hyperbolic motion describes the solution of relative motion of a pair of electric charges with the same sign, and it is interesting to study problems such as scattering by dispersion. In this case, also in the periapsis region, the secondary is affected by major forces; then, in this region, the density of points is lower if the natural time is used.

Due to the previous reason, the local truncation error depends strongly on the orbital region when we use numerical integration methods. In order to achieve a more uniform distribution of these errors on the orbit, there are three main techniques:

1. The use of a very small stepsize.
2. The use of an adaptive stepsize method.
3. The use of a change in the temporal variable to arrange an appropriate distribution of the points on the orbit so that the points are mostly concentrated in the regions where the acceleration and curvature are maxima.

This paper follows the third technique. Several authors have already studied this question for the elliptic motion, starting from the Sundman transformation [2], introducing a new temporal variable τ related to the time t through $dt = Crd\tau$. Other transformations are proposed by Nacozy [3], Brumberg [4] proposed the use of the regularized length of arc and Brumberg and Fukushima [5] introduced the elliptic anomaly as temporal variable. Janin [6,7] and Velez [8] generalized Sundman transformations $dt = C_\alpha r^\alpha d\tau_\alpha$, Ferrandiz [9] introduces the generalized elliptic anomaly, López [10] introduces a new family of anomalies, called natural anomalies and López [11] defines a geometrical family of transformations that includes the true anomaly f, the eccentric anomaly g and the antifocal anomaly f'. These transformations are defined as $dM = Q(r)d\Psi$ where $M = n(t - t_0)$ is the mean anomaly, $n = \sqrt{|\mu|/a^3}$ the fictitious mean motion t the time, t_0 the epoch of periapsis transit, Ψ the new anomaly derived from the change in the temporal variable and $Q(r)$ is a function of the vector radius r called partition function.

The geometrical family of continuous transformations [11] for the elliptic case presents good properties such as closed formulas for the most common quantities of the two body problem, closed form in the coefficients of Fourier developments used in the analytical theories of the perturbed motion and an appropriate performance in the numerical methods. In this paper, we try to extend this family of transformations to hyperbolic attractive and repulsive motion in order to obtain the most important quantities of the two body problem with respect to these new variables. This work includes an initial study of the dependence on the new variables of the integration errors in numerical methods.

The rest of this paper is organized as follows: In this section, the general background has been introduced. In Section 2, the properties of a generalized geometric family of the anomalies will be described. In particular, we will obtain the most common quantities of the problem in closed form using an arbitrary anomaly from this family for the hyperbolic attractive case. In Section 3, in a similar way, we extend the analytical study to hyperbolic repulsive case obtaining closed formulas. In Section 4, a set of numerical examples about the attractive hyperbolic two body problem will be considered. In Section 5, the main conclusions and remarks will be exposed.

2. Generalized geometric anomalies

In this section, a new family of anomalies depending on one parameter is defined. We represent in Fig. 1 the hyperbolic orbit corresponding to the motion of the gravitational two-body problem. This hyperbola is defined by its major semiaxis

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