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Surface parameterization based on polar factorization

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ABSTRACT

A mapping between measurable subsets of Euclidean space can be uniquely factorized to the composition of a measure-preserving mapping and an optimal transportation map, the later is also the gradient map of a convex function. This work introduces an algorithm based on variational approach to compute this type of polar factorization for mappings between planar domains and some direct applications. Our method greatly increases the flexibility for surface parameterizations by balancing between area distortion and angle distortion, and improves the accuracy and numerical stability for down steam geometric processing tasks.

1. Introduction

1.1. Motivation

Surface parameterization refers to the process of mapping the surface onto a planar domain smoothly and bijectively. It plays an important role in digital geometry processing, and has broad applications, such as texture mapping, spline fitting and surface registration.

In practice, it is highly desirable to minimize the distortions induced by the parameterization. There are mainly two types of distortions, angle distortion and area distortion. A mapping without both angle and area distortion must be isometric, therefore the Gaussian curvature will be kept. This is impossible, unless the original surface is developable.

For general surfaces, conformal geometric methods have been developed to pursue parameterizations without angle distortions. Recently, algorithms based on optimal mass transportation have been introduced to achieve parameterizations without area distortions.

Conformal parameterizations have no angle distortions, but they may induce large area distortions. For cylindrical shapes, the area distortions can be exceptionally large in terms of the cylinder height. The huge area distortions cause severe numerical instability and aliasing in rendering. On the other hand, area-preserving parameterizations have no area distortions, but they may cause huge angle distortions. In digital geometry processing, many geometric tasks boil down to solve geometric partial differential equations on the surface. The geometric PDEs are converted to sparse linear systems using Finite Element Method. The numerical stability of the linear systems heavily depends on the angle structure of the discrete

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triangular mesh. If the triangulation has too many obtuse angles, the linear system is highly unstable, and the computational results are not reliable.

Therefore, the parameterization with a good balance between angle distortion and area distortion is highly preferred. The polar factorization of general surface mapping method has great promise to tackle this challenging problem. Given a diffeomorphism φ, it can be decomposed into the composition of two mappings

$$\varphi = \nabla u \circ s, \tag{1.1}$$

where s is area-preserving, u is a convex function and the gradient map ∇u deforms the area in the most economical way (an optimal mass transportation map). The convex function u and the optimal mass transportation map ∇u are solely determined by the source and the target mass density functions. This gives a practical way to control the area distortion.

Suppose the initial map φ is conformal, by varying the convex function u, we can deform φ to s, namely we build a path φ_t in the mapping space, connecting the angle preserving mapping $\varphi_1 = \varphi$ to the area preserving mapping $\varphi_0 = s$. By choosing the parameter t, one can find the optimal parameterization φ_t, most appropriate for the application.

1.2. Optimal mass transport

In 1781, Gaspar Monge concerned finding the optimal way, in the sense of minimal transportation cost, of moving a pile of soil from one site to another [1]. The problem was given a modern formulation in the work of Kantorovich [2], and so is now known as the Monge–Kantorovich problem.

Given two domains in $\mathbb{R}^d$ with smooth boundaries, (Ω_0, μ_0) and (Ω_1, μ_1) with mass densities μ_0 and μ_1 respectively. Let $\varphi : \Omega_0 \to \Omega_1$ be a measure-preserving map, namely, for any Borel set $B \subset \Omega_1$, the total mass in B given by $\int_B \mu_1(y)dy$ equals the mass of its preimage $\varphi^{-1}(B)$, $\int_{\varphi^{-1}(B)} \mu_0(x)dx$. The L^p transportation cost of φ is given by

$$E(\varphi) := \int_{\Omega_0} \|x - \varphi(x)\|^p \mu_0(x)dx.$$

The Monge–Kantorovich problem is to find the optimal mapping, which minimizes the transportation cost. The existence and the uniqueness of the optimal transport plan were proved by Kantorovich [2] in the 1940s.

At the end of 1980s, Brenier [3] proved that the optimal transport map is the gradient map of a convex function, when the transportation cost is a quadratic function of the Euclidean distance. Essentially, there is a unique convex function $u : \Omega_0 \to \mathbb{R}$, its gradient map $\nabla u : x \mapsto \nabla u(x)$ is measure preserving and is the optimal transportation map.

1.3. Polar factorization

The mapping polar factorization can be treated as the generalization of matrix polar decomposition and vector field Helmholtz decomposition. Suppose (Ω_0, μ_0) and (Ω_1, μ_1) are subdomains in the Euclidean space $\mathbb{R}^d$, $\varphi : \Omega_0 \to \Omega_1$ is a diffeomorphism between them. Then there is a convex function $u : \Omega_0 \to \mathbb{R}$, whose gradient map $\nabla u : x \mapsto \nabla u(x)$ maps from Ω_0 to Ω_1. Furthermore, there is a volume-preserving mapping $s : \Omega_0 \to \Omega_0$, such that φ can be decomposed to $\varphi = \nabla u \circ s$. This decomposition is unique. As shown in Fig. 1, a conformal mapping φ in (c) is decomposed to an area preserving mapping from (a) to (b) and a gradient map from (b) to (c).

If the source domain coincides with the target domain, denoted as Ω, then all the volume-preserving diffeomorphisms form a Lie group $S(\Omega)$, which is non-convex. Given a diffeomorphism $\varphi : \Omega \to \Omega$ with polar decomposition $\varphi = \nabla u \circ s$, s is the unique L^2 projection onto $S(\Omega)$. Assume Ω has a volume form (measure) μ, then φ induces a push forward measure $\varphi_{\#}\mu$, ∇u is the unique optimal mass transportation map from (Ω, μ) to $(\Omega, \varphi_{\#}\mu)$.

1.4. Contributions

This work develops practical algorithms

1. A variational method for polar factorization for surface mappings.
2. An efficient method for one parameter family of surface parameterizations $\{\varphi_t\}$, where φ_0 is area-preserving, the area-distortion of φ_t increases monotonously with respect to t, and φ_1 is angle-preserving.

This method greatly increases the flexibility for surface parameterizations by balancing between area distortion and angle distortion, and improves the accuracy and numerical stability for down steam geometric processing tasks.

2. Related work

In this section, we briefly review the most related works.

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