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A patch-based low-rank tensor approximation model for multiframe image denoising

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HIGHLIGHTS

- An algorithm for low-rank tensor approximation is proposed.
- The algorithm is based on matrix factorization to all-mode unfoldings of the tensor.
- The algorithm is embedded in a patch-based multiframe image denoising method.
- The performance of the denoising method is competitive in the numerical experiments.

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ABSTRACT

Compared with matrix, tensor is a more natural representation for multiframe image, such as hyperspectral image and MRI image. Low-rankness of tensor is essential to describe the intrinsic geometrical structure of these data. Patch-based low-rank models have shown their ability to exploit spatial redundancy of computer vision data especially for natural image denoising. However, most of the existed patch-based matrix models are based on two dimensional low-rankness, which cannot fully reveal the correlation of every direction in high-order multiframe images; the existed patch-based tensor models either need additional assumptions or need SVD in every loop of iteration which is computationally expensive. In this paper, we propose a novel patch-based model to recover a low-rank tensor by simultaneously performing low-rank matrix factorizations to the all-mode matricizations of the underlying low-rank tensor. An augmented Lagrangian alternating minimization algorithm is implemented to solve the model along with two adaptive rank-adjusting strategies when the exact rank is unknown. We apply the proposed algorithm to multiframe image denoising by exploiting the nonlocal self-similarity. Experimental results show that our algorithm can better preserve the sharpness of important image structures and outperforms several state-of-the-art denoising methods.

1. Introduction

Low-rankness has attracted increasing research interest because of its ability to reveal the intrinsic geometrical structure of data. Low-rank matrix approximation, aims to recover the underlying low rank matrix from its degraded observation. It has been applied in diverse areas of machine learning and computer vision. For instance, it is part of the most critical concepts for subspace segmentation [1], moving objects extraction from surveillance video [2], and advertisement recommendation [3]. With the development of convex and non-convex optimization methods, many important models and algorithms on low-rank matrix approximation have been reported in recent years [4–8].

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Tensor is a generalization of *matrix*, as *matrix* is a generalization of *vector*. Low-rank matrix approximation, which can be considered as a special case of low-rank tensor approximation when the order of tensor is two. To reconstruct a low-rank tensor from its degraded observation, one can unfold it into a matrix and apply a low-rank matrix approximation algorithm. However, this kind of method does not fully utilize the information contained in the tensor. While the better performance of [9,10] encourage us to take advantages of all mode low-rankness.

Patch-based model for image processing has attracted much attention. This category of models takes advantage of image nonlocal self-similarity (NSS) [11] to improve the traditional process, leading to effective performance in natural image restoration. The NSS prior refers to the fact that for a given local patch in a natural image, one can find many similar patches to it across the image. Many effective patch-based low-rank matrix approximation algorithms have been proposed to improve the denoising process, such as [12,13,9,14]. Patch-based low-rank tensor approximation algorithms for image denoising have been also proposed as a natural generalization. Some of them demonstrate very competitive results [15–18].

In this paper, we proposed a novel low-rank tensor approximation algorithm founded on the decomposition of low-rank tensor. Although it is non-convex, the performance of its matrix situation give us confidence. Before introducing our model and algorithm, we review some notation and preliminary knowledge of tensor first.

1.1. Notation and preliminary knowledge

Following [19], we use bold upper-case letters $\mathbf{X}, \mathbf{Y}, \ldots$ to denote matrices, bold calligraphic letters $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}}, \ldots$ to denote tensors, and $x_{i_1 \ldots i_N}$ to denote the $(i_1, \ldots, i_N)$th component of an N-way tensor $\boldsymbol{\mathcal{X}}$. Similarly as the situation of vector and matrix, the inner product of two tensors $\boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is defined as

$$\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{Y}} \rangle = \sum_{i_1=1}^{I_1} \cdots \sum_{i_N=1}^{I_N} x_{i_1 \ldots i_N} y_{i_1 \ldots i_N}.$$

With the definition of inner product, the F-norm (Frobenius norm) of a tensor $\boldsymbol{\mathcal{X}}$ is defined as $\|\boldsymbol{\mathcal{X}}\|_F = \sqrt{\langle \boldsymbol{\mathcal{X}}, \boldsymbol{\mathcal{X}} \rangle}$. By fixing all indices of a tensor $\boldsymbol{\mathcal{X}}$ except one, we can get a vector which is called a *fiber* of $\boldsymbol{\mathcal{X}}$. The mode-n *matricization* (which is also called *unfolding*) of $\boldsymbol{\mathcal{X}} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is denoted as $\mathbf{X}_{(n)} \in \mathbb{R}^{I_n \times \Pi_{j \neq n} I_j}$, which is a matrix whose columns are the mode-n fibers of $\boldsymbol{\mathcal{X}}$ arranged in the lexicographical order. We define the n-rank of an N-way tensor $\boldsymbol{\mathcal{X}}$ as an N dimensional vector in this paper: $\text{rank}(\boldsymbol{\mathcal{X}}) = (\text{rank}(\mathbf{X}_{(1)}), \ldots, \text{rank}(\mathbf{X}_{(N)}))$. This definition is related to the Tucker decomposition [20] which can be viewed as a generalization of SVD. There is another popularly used definition which is based on the CANDECOMP/PARAFAC decomposition [21]. In this paper, we say $\boldsymbol{\mathcal{X}}$ is (approximately) low-rank if $\mathbf{X}_{(n)}$ is (approximately) low-rank for all n. Here, an approximately low-rank matrix means the although matrix is not exactly a low-rank one, most of its singular values are far less than the its largest ones. We say the tensor is approximately low-rank when all of its unfoldings are exactly low-rank or approximately low-rank.

1.2. Related work

Low-rank matrix approximation algorithms can be roughly summarized into two categories: the nuclear norm minimization (NNM) related algorithms and the low rank matrix factorization (LRMF) algorithms. The nuclear norm of a matrix $\mathbf{X}$ is the sum of its singular values, which is denoted as $\|\mathbf{X}\|_* = \sum_i |\sigma_i(\mathbf{X})|_1$, NNM aims to approximate a low-rank matrix by its approximation via minimizing the nuclear norm of the approximation. NNM is the tightest convex relaxation of the original non-convex problem with certain regularization term. It has been proved that most low rank matrices can be perfectly recovered by solving an NNM problem in [22], and numerous algorithms are proposed based on this idea [4,2,1]. LRMF aims to find an approximation close enough to the matrix under certain condition, while being able to be factorized into the product of two low rank matrices. A variety of LRMF methods have been proposed [23,24,5,25]. The LRMF problem is basically a nonconvex optimization problem.

Different from the natural sparsity measure (rank) for matrices, it is more complicated to construct a rational low-rankness measure to describe the intrinsic correlations along various tensor modes. [26] promoted sum of ranks measure by relaxing it to a tighter convex form. [9] measured sparsity of a tensor with the sum of the ranks of all unfolding matrices along all modes and relaxing with trace norms. [27] applied the same sparsity measure and solve the problem by ADMM. [28] measured the sparsity of a tensor by its largest rank of all unfolding matrices, and relaxed it with a sum of exponential forms. [29] developed a tensor-SVD based sparsity measure mainly for videos. [30] designed a measurement by the size of the fundamental Kronecker basis. [10] investigated the factorization of all mode matricization of tensor and had remarkable performance.

Patch-based denoising lies at the heart of most denoising algorithms. Patch-based methods first proposed in [11], in that paper, the authors explore the non-local self-similarity of natural images. Motivated by this idea, numerous algorithms have been proposed. As to low-rank tensor approximation, the patch-based methods [15,18,16,17] have carried out very valuable attempt. The good performance of these algorithms has inspired us to adopt the patch-based idea to the application of our algorithm.