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A new modified weak Galerkin finite element scheme for solving the stationary Stokes equations

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ABSTRACT

In this paper, a modified weak Galerkin method is proposed for the Stokes problem. The numerical scheme is based on a novel variational form of the Stokes problem. The degree of freedoms in the modified weak Galerkin method is less than that in the original weak Galerkin method, while the accuracy stays the same. In this paper, the optimal convergence orders are given and some numerical experiments are presented to verify the theory.

1. Introduction

In this paper, we consider the Stokes problem

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\mathbf{u} = \mathbf{g}, \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d (d = 2, 3)$. The right-hand side $\mathbf{f} \in [L^2(\Omega)]^d$ is the source term, and $\mathbf{g}$ is the boundary condition satisfying

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} = 0,$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$. Without loss of generality, in this paper we consider the homogeneous Dirichlet boundary condition, i.e. we assume $\mathbf{g} = \mathbf{0}$. In the past several decades, many efforts have been dedicated to the

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numerical simulation of the Stokes problem. A variety of numerical methods, such as finite difference method, finite element method [1], and discontinuous Galerkin method [2], have been applied to the Stokes problem. The details of classical finite element method for the Stokes problem are introduced in [3] and the references therein.

Recently, a new type of numerical method, called the weak Galerkin method, has been developed for partial differential equations [4]. The main idea of weak Galerkin method is to use totally discontinuous piecewise polynomials as basis functions, and use specifically defined weak derivatives instead of classical derivative operators in the numerical scheme. The weak Galerkin method is efficient for many types of partial differential equations, such as second order elliptic equation [5,4], biharmonic equation [6,7], Brinkman equation [8,9], and Maxwell equation [10], etc. Some numerical techniques, such as posterior estimate [11], have also been applied to the weak Galerkin method.

For the Stokes problem (1.1)–(1.3), the classical variational form is to find $\mathbf{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$, such that

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) - (\nabla \cdot \mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d, \tag{1.4}$$

$$(\nabla \cdot \mathbf{u}, q) = 0, \quad \forall q \in L_0^2(\Omega). \tag{1.5}$$

The variational problem (1.4)–(1.5) can be solved by the conforming finite element method, and many kinds of elements satisfying stable condition are proposed [3]. The corresponding weak Galerkin method [12,13] are also developed for the variational problem (1.4)–(1.5). However, there is another variational form as follows. Find $\mathbf{u} \in [H_0^1(\Omega)]^d$ and $p \in L_0^2(\Omega)$ on $\partial\Omega$, and

$$(\nabla \mathbf{u}, \nabla \mathbf{v}) + \langle \nabla p, \mathbf{v} \rangle = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in [H_0^1(\Omega)]^d, \tag{1.6}$$

$$\langle \nabla q, \mathbf{u} \rangle = 0, \quad \forall q \in L_0^2(\Omega), \tag{1.7}$$

where ∇p and ∇q are functionals defined by distributions. Due to its flexibility, the weak Galerkin method can be applied to variational problem (1.6)–(1.7), which is unusual for the conforming finite element method. The reason that we study a WG scheme based on the gradient–gradient variational form (1.6)–(1.7) is mainly coming from that this scheme is suit for the mixed form of Darcy which would present a better approximation for this case. In fact, for the complex porous media with interface conditions, people could use Stokes–Darcy interface model to describe this problem. In order to provide a more efficient weak Galerkin scheme, we prefer to utilize this gradient–gradient weak form to approximate the model. To unify the weak form of this interface problem, we present the corresponding numerical analysis results of this gradient–gradient form for Stokes problem.

For the Stokes problem, the advantages of weak Galerkin method are two folds. One is that the weak Galerkin method is valid on regular polytopal meshes. The other one is that the weak Galerkin method is of high flexibility. Most types of elements satisfy the inf–sup condition naturally in the weak Galerkin method, and we do not need to construct finite element spaces specifically. The main problem of the weak Galerkin method is the calculation cost. Other than the freedoms on each element, the weak Galerkin method introduces freedoms on each edge/face in the partition, so the degree of freedom is much more higher than the classical finite element method. To this end, a modified weak Galerkin (MWG) method is proposed [14,15]. The main idea is to use the average of $\mathbf{u}_0$ to replace $\mathbf{u}_b$, thus we can reduce the degree of freedoms without loss of accuracy.

In this paper, we shall apply the modified technique to the weak Galerkin method introduced in [16]. We shall prove that by using the modified technique, the degree of freedoms decreases while the convergence rates stay the same. The rest of the paper is structured as follows. In Section 2, we shall propose the modified weak Galerkin scheme. Section 3 is devoted to the stability analysis. In Section 4, we give the error equation, and the error estimates are proved in Section 5. Some numerical experiments are presented in Section 6.

2. Modified weak Galerkin method

First, we introduce some preliminaries and notations. In this paper, we use the notations of standard Sobolev space. For any domain K and real number s, denote by $\|\cdot\|_{s,K}$ and $(\cdot,\cdot)_{s,K}$ the norm and inner-product in $H^s(K)$ space. When $s = 2$ and $K = \Omega$, we shall drop the subscript. When K is an edge/face, we also use $\langle\cdot,\cdot\rangle_{s,K}$ to represent the inner-product.

Suppose $\mathcal{T}_h$ is a polytopal partition of Ω satisfying the regular assumptions verified in [17]. For each element $T \in \mathcal{T}_h$, denote by h_T the diameter of T, and $h = \max_{T\in\mathcal{T}_h}$ is the mesh size of $\mathcal{T}_h$. We also use $P_k(T)$ to represent the space of polynomials on element T whose degree is no more than k. Denote by $\mathcal{E}_h$ the union of all edges/faces in $\mathcal{T}_h$, and $\mathcal{E}_h^0$ the union of all interior edges/faces in $\mathcal{T}_h$. We also use $P_k(e)$ to represent the space of polynomials on edge/face e whose degree is no more than k. In this paper, C represents a generic constant independent of h.

In [16], we defined the weak Galerkin finite element spaces as follows:

$$\mathcal{V}_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}, \mathbf{v}_0 \in [P_k(T)]^d, \mathbf{v}_b \in [P_k(e)]^d, \forall T \in \mathcal{T}_h, e \in \mathcal{E}_h, \mathbf{v}_b = \mathbf{0} \text{ on } \partial\Omega\},$$

and

$$W_h = \left\{ q = \{q_0, q_b\}, q_0 \in P_{k-1}(T), q_b \in P_k(e), \forall T \in \mathcal{T}_h, e \in \mathcal{E}_h, \int_\Omega q_0 = 0 \right\}.$$