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The number of regular control surfaces of toric patch

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ABSTRACT

Through a rational map, a toric patch is defined associated to a lattice polygon, which is the convex of a given finite integer lattice points set $\mathcal{A}$. The classical rational Bézier curves, rational triangular and tensor-product patches are special cases of toric patches. One of the geometric meanings of toric patch is that the limiting of the patch is its regular control surface, when all weights tend to infinity. In this paper, we study the number of regular decompositions of $\mathcal{A}$, and the relationship between regular decompositions and the corresponding secondary polytope. What is more, we indicate that the number of regular control surfaces of toric patch associated with $\mathcal{A}$ is equal to the number of regular decompositions of $\mathcal{A}$.

1. Introduction

Bézier curves/patches are effectively used in many areas of Computer Aided Geometric Design (CAGD), Computer Aided Design (CAD) and Geometric Modeling (GM) [1,2]. They were first described in 1962 by the French engineer Pierre Bézier, who used them to design automobile bodies [3]. Although the Bézier curves/patches have good explicit formula and properties, as polynomial curves/patches, the representation of some basic curves/patches is still lacking, such as conics. So, the rational Bézier curves/patches appeared. The rational Bézier curves/patches share the most of geometric properties with the polynomial ones. The main difference is that the shape of a rational Bézier curve/patch can be not only controlled by its control structure but also controlled by its weights. Especially, the enough large weight pulls the curve/patch towards the corresponding control point [2], which is called the geometric meaning of a single weight. The properties of Bézier curves/patches can be found in Farin's book [1], and the geometric algorithms refer to the paper [4].

Toric variety was introduced and developed from the early 1970s [5]. The theory of toric variety is mainly from algebraic geometry [5] and combinatorics [6]. In algebraic geometry, a toric variety is an algebraic variety containing an algebraic torus as an open dense subset, such that the action of the torus on itself extends to the whole variety. The geometry of a toric variety is fully determined by the combinatorics of its associated fan, which often makes computations far more tractable.

In 1992, it is probable that Warren [7] was the first who noticed the real toric variety can be applied in CAGD. After that, in 2002, Krasauskas [8] defined a kind of rational multisided patch—toric patch (or toric surface patch), which is associated with a finite integer lattice points set. When all of the points of the finite integer lattice points set lie on a line segment, we can get the classical rational Bézier curve. In this paper, this kind of curve (rational Bézier curve) is also called the toric Bézier curve, since it is the special case of toric patch in fact. What is more, the classical Bézier triangle, tensor-product

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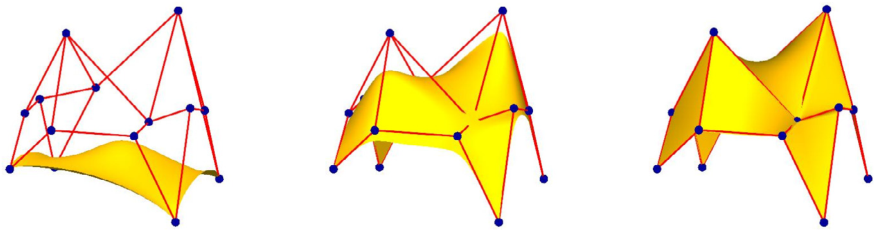

Fig. 1. Bicubic Bézier patch and its regular control surface.

Bézier patch, and Warren's hexagonal patch [7] are special cases of the toric patches, while the corresponding polygons are triangle, rectangle and hexagon [8].

In geometric modeling, the control structure of toric patch is well-defined and has geometric significance. The patch is controlled by the control structure defined by the corresponding control points, and controlled by the weights, i.e., enough large weights pull the patch towards the corresponding control structure. In 2011, García-Puente, Sottile and Zhu [9] explained the limiting surface of toric patch when all weights tend to infinity, which is called the regular control surface, and generalized the geometric meaning of a single weight of rational Bézier curve/patch [2]. For example, Fig. 1 shows a bicubic rational Bézier patch approaching to its regular control surface, when all weights tend to infinity. Notice that different lifting functions may induce different regular control surfaces, which are all the possible limiting positions of the same toric patch [9]. Then a natural question is *how many regular control surfaces of a toric patch?* In this paper, we shall answer this question and present method to compute the number of regular control surfaces of toric patches (including rational Bézier curves, rational Bézier triangles and tensor-product Bézier patches) by the theory from combinatorics and CAGD.

This paper is organized as follows. In Section 2, we study on the relationship between regular decompositions and the secondary polytope. We give an explicit formula to compute the number of regular decompositions of a given finite points set. In Section 3, we show how to compute the number of regular control curves of the rational Bézier curve through a representative example. Furthermore, we indicate that the number of regular control surfaces of a toric patch is equal to the number of regular decompositions of $\mathcal{A}$ in Section 4. Section 5 concludes the whole paper.

2. Regular decompositions and regular control surfaces

Let $\mathcal{A} = \{a_1, a_2, \ldots, a_m\} \subset \mathbb{Z}^{n-1}$ be a finite integer lattice points set (or a m point configuration), and convex polytope $\Delta = Conv(\mathcal{A})$. A continuous function [10] $g : \Delta \to \mathbb{R}$ is called concave, if for any $a_1, a_2 \in \Delta$, we have $g(ta_1 + (1-t)a_2) \geq tg(a_1) + (1-t)g(a_2)$, $0 \leq t \leq 1$. Furthermore, the domain of function g is linear. Suppose $\mathcal{T}$ is a triangulation of $\mathcal{A}$. The function g is called $\mathcal{T}$-piecewise-linear if it is affine-linear on every simplex induced by $\mathcal{T}$.

Definition 1 (*[10]*)**.** A triangulation $\mathcal{T}$ of $\mathcal{A}$ is called regular if there exists a concave $\mathcal{T}$-piecewise-linear function whose domains of linearity are precisely (maximal) simplices induced by $\mathcal{T}$.

In book [10], Gel'fand, Kapranov and Zelevinsky provided the relationship between regular triangulations and the secondary polytope. Let $\mathcal{T}$ be a triangulation of $\mathcal{A}$, and σ is one of the simplices of Δ induced by $\mathcal{T}$. We assume that $dim(\Delta) = n-1$, and σ is a $(n-1)$-simplex. $Vol(\sigma)$ denotes the volume of the σ. And $\varphi_{\mathcal{T}}(a_i) \in \mathbb{R}^{n-1}$ are the vertices of secondary polytope which is defined as follows [11]:

$$\varphi_{\mathcal{T}}(a_i) = \sum_{i=1}^{m} \left(\sum_{a_i \in \sigma} \{Vol(\sigma) : \sigma \in \Delta \text{ and } a_i \in \sigma\} \right) \cdot e_i, \tag{1}$$

where the summation is over all (maximal) simplices for which a_i is a vertex, e_i is the ith standard basis vector of $\mathbb{R}^{n-1}$. In particular, $\varphi_{\mathcal{T}}(a_i) = 0$ if a_i is not a vertex of any simplex induced by $\mathcal{T}$. *The secondary polytope* $\Sigma(\mathcal{A})$ is the convex hull with the volume vectors $\varphi_{\mathcal{T}}$ of all regular triangulations of $\mathcal{A}$. The following result shows the combinatorial relationship of regular triangulations and the secondary polytope $\Sigma(\mathcal{A})$. Furthermore, it is an important result in geometric combinatorics and real algebraic geometry.

Theorem 1 (*[10,11]*)**.** *The vertices of the secondary polytope $\Sigma(\mathcal{A})$ of a point configuration $\mathcal{A}$ are in one-to-one correspondence with the regular triangulations of $\mathcal{A}$.*

What is more, we are able to compute the number of regular triangulations of $\mathcal{A}$ by the construction of the secondary polytope by this result. And using formula (1), we could find the precise positions of vertices of $\Sigma(\mathcal{A})$.

A *polyhedral subdivision* of a convex polytope $\Delta \subset \mathbb{R}^{n-1}$ is a decomposition of Δ into finite number of polyhedral complexes such that the intersection of any two of these polyhedral complexes is a common face of them both (maybe empty). For a polyhedral subdivision of Δ, we mean simply a polyhedral subdivision of Δ into polyhedral complexes with

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