Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The maximum number and its distribution of singular points for parametric piecewise algebraic curves[☆]

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ARTICLE INFO

Article history:
Received 5 September 2016
Received in revised form 30 November 2016

Keywords:
Parametric piecewise algebraic curve
Singular points
Semi-algebraic set

ABSTRACT

The piecewise algebraic curve, as the zero set of a bivariate spline function, is a generalization of the classical algebraic curve. Based on the previous method presented by Lai et al. (2009), we show that computing singular points of parametric piecewise algebraic curves amounts to solving parametric piecewise polynomial systems. In this article, we give a method to compute the maximum number and its distribution of singular points for a given parametric piecewise algebraic curve. This method also produces necessary and sufficient conditions of its parameters must be satisfied. An illustrated example shows that our method is flexible.

1. Introduction

We denote by $\mathbb{R}[x, y]$ the polynomial ring in variables x and y over real number field $\mathbb{R}$ and $\mathbb{P}_k[x, y]$ the set of bivariate polynomials in $\mathbb{R}[x, y]$ with real coefficients and total degree $\leqslant k$. A polynomial $f(x, y) \in \mathbb{R}[x, y]$ is called an irreducible polynomial if the polynomial f cannot be divided by any other polynomial except a constant or itself. Algebraic curve

$$\mathcal{Z}(f) := \{(x, y) \in \mathbb{R}^2 \mid f(x, y) = 0,\ f(x, y) \in \mathbb{P}_k[x, y]\}$$

is called an irreducible algebraic curve if $f(x, y)$ is an irreducible polynomial.

Let Ω be a connected and bounded region in $\mathbb{R}^2$. Using a finite number of irreducible algebraic curves in $\mathbb{R}^2$, we divide the region Ω into several simply connected regions, which are called the partition cells. Denote by Δ the partition of the region Ω, let $\delta_1, \delta_2, \ldots, \delta_T$ be a given ordering of the cells in Δ, and let $\Omega = \bigcup_{i=1}^{T} \delta_i$. Also, we write $\Delta = \{\delta_1, \delta_2, \ldots, \delta_T\}$ and the interior of each cell δ_i can be described as

$$\delta_i = \{(x, y) \in \mathbb{R}^2 \mid g_1^{[i]}(x, y) > 0, g_2^{[i]}(x, y) > 0, \ldots, g_{N_i}^{[i]}(x, y) > 0\},$$

where $g_1^{[i]}(x, y), \ldots, g_{N_i}^{[i]}(x, y) \in \mathbb{R}[x, y]$ are irreducible polynomials.

Denoted by $C^\mu(\Delta)$ the set of C^μ functions $s(x, y)$ on Ω. Assume the restriction $s(x, y)|_{\delta_i}$ is a polynomial which belongs to $\mathbb{R}[x, y]$ and let

$$\mathbb{P}_k(\Delta) = \{s(x, y)|s(x, y)|_{\delta_i} \in \mathbb{P}_k[x, y],\ i = 1, 2, \ldots, T\}$$

be the set of piecewise polynomials defined on Δ with total degree $\leqslant k$.

☆ This work was supported by the National Natural Science Foundation of China (Nos.11401526, 11101366, 11671068 and 11271328), the Zhejiang Provincial Natural Science Foundation of China (No.LY14A010001).

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http://dx.doi.org/10.1016/j.cam.2017.01.023
0377-0427/

The bivariate spline space with smoothness μ and degree k over Ω with respect to Δ is defined as follows:

$$S_k^\mu(\Delta) = \{s(x, y) \mid s(x, y) \in C^\mu(\Delta) \cap \mathbb{P}_k(\Delta)\}.$$

The zero set

$$\mathcal{Z}(s) := \{(x, y) \in \Omega \mid s(x, y) = 0,\ s(x, y) \in S_k^\mu(\Delta)\}$$

is called a piecewise algebraic curve.

Obviously, the piecewise algebraic curve is a generalization of the classic algebraic curve. However, it is difficult to study piecewise algebraic curve not only because of the complexity of the partition but also because of the possibility of $\{(x, y) \mid s(x, y) = 0\} \cap \delta_i = \emptyset$.

Piecewise algebraic curve is originally introduced by Wang in the study of multivariate spline interpolation. He pointed out that the given interpolation knots are properly posed if and only if they are not lie in a non-zero piecewise algebraic curve [1]. In recent years, Wang and his researcher group have done a lot of significant work on piecewise algebraic curves (see [1–19]). For example, the Bezout theorem [3,4,13,16], Nöther-type theorem [7,9,10], Cayley–Bacharach theorem [8,14] and Riemann–Roch type theorems [9] of piecewise algebraic curves were established. Besides, piecewise algebraic curve also relates to the remarkable Four-Color conjecture [4]. Moreover, Lai et al. [11,12] discussed the real zeros of the zero-dimensional parametric piecewise algebraic variety and produced the corresponding necessary and sufficient conditions of coefficients. Also, the Viro method for construction of Bernstein–Bézier algebraic hypersurface piece and piecewise algebraic curve are discussed [17,18]. In sum, the piecewise algebraic curve is a new and important topic in Computer Aided Geometry Design and Computational Geometry and has many applications in various fields. It is necessary and significant to study the related problems on piecewise algebraic curves.

The singular points of a given real algebraic curve $\mathcal{Z}(f)$ ($SPT(f)$ for short) are those points on the curve where both partial derivatives vanish, i.e.,

$$SPT(f) = \left\{(x, y) \mid f(x, y) = f_x(x, y) = f_y(x, y) = 0\right\}.$$

The singular points and its classification for a given algebraic curve is a basic and important topic in algebraic geometry [20,21]. The detection of singular points helps to determine the geometrical shape and topology of a real algebraic curve. In 1990, Sakkalisa et al. [21] described an algorithm for determining whether an irreducible algebraic curve $f(x, y) = 0$ with total degree $\geqslant 3$ is singular, and if so, isolating its singular points and computing their multiplicities. For a given plane rational curve in parametric form, there are several efficient methods to detect and compute the singular points [22].

Hence, it is of theoretical and practical significance to study singular points of a given parametric piecewise algebraic curve, especially the maximum number and its distribution of its singular points. Certainly, it helps us to determine the topological structure of piecewise algebraic curves better.

Now, we firstly give the definition of parametric piecewise algebraic curves. Let $V = (v_1, v_2, \ldots, v_r)$ be the parameters and denoted by $\mathbb{R}[V][x, y]$ the set of all polynomials in variables x and y with coefficients in $\mathbb{R}[V]$. Obviously, $p \in \mathbb{R}[V][x, y]$ can be viewed as $p = p(V, x, y)$.

Definition 1.1. Let $s = s(V, x, y) \in S_k^\mu(\Delta)$ and $s|_{\delta_i} \in \mathbb{R}[V][x, y]$ for each cell δ_i in Δ, where $V = (v_1, v_2, \ldots, v_r)$ and (x, y) are viewed as parameters and variables, respectively. Then $\mathcal{Z}(s)$ is called a parametric piecewise algebraic curve.

In other words, piecewise algebraic curve is "parametric" means it contains symbolic coefficients and also allows to have some certain constant coefficients.

From the theory of smoothing cofactor of bivariate spline [1], we can easily have

Lemma 1.1. *Let $s = s(V, x, y) \in S_k^\mu(\Delta)$ and $s|_{\delta_i} \in \mathbb{R}[V][x, y],\ i = 1, 2, \ldots, T$. If the point $p \in \mathcal{Z}(s)$ and $p \in \delta_i \cap \delta_j,\ i, j \in \{1, 2, \ldots, T\}$, then p is the singular (regular) point of algebraic curve $\mathcal{Z}(s|_{\delta_i}) = \{(x, y) \in \delta_i \mid s|_{\delta_i}(V, x, y) = 0\}$ if and only if p is the singular (regular) point of algebraic curve $\mathcal{Z}(s|_{\delta_j}) = \{(x, y) \in \delta_j \mid s|_{\delta_j}(V, x, y) = 0\}$.*

Proof. By using smoothing cofactor of bivariate spline [1], we have

$$s|_{\delta_i}(V, x, y) - s|_{\delta_j}(V, x, y) = l_{ij}(x, y)^{\mu+1} q_{ij}(V, x, y),$$

where $\mathcal{Z}(l_{ij})$ is the common algebraic curve of $\delta_i \cap \delta_j$, and $q_{ij}(V, x, y) \in \mathbb{P}_d[x, y],\ d = k - (\mu + 1)\deg(l_{ij})$. From simple computation, the result is proved.

Therefore, singular points of a given parametric piecewise algebraic curve are defined as follows:

Definition 1.2. Let $s = s(V, x, y) \in S_k^\mu(\Delta)$. If $p \in \mathcal{Z}(s)$ and there exists $i \in \{1, 2, \ldots, T\}$ such that $p \in \overline{\delta_i}$ (*the closure of* δ_i) and p is the singular point of algebraic curve $\mathcal{Z}(s|_{\overline{\delta_i}})$, then p is called a singular point of parametric piecewise algebraic curve $\mathcal{Z}(s)$. Otherwise, p is called a regular point of $\mathcal{Z}(s)$.

Please cite this article in press as: J. Wu, C. Zhu, The maximum number and its distribution of singular points for parametric piecewise algebraic curves, Journal of Computational and Applied Mathematics (2017), http://dx.doi.org/10.1016/j.cam.2017.01.023