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The Use of the Dyadic Partition in Elementary Real Analysis

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Abstract

In [3], Gordon presented alternate proofs of several well known results in elementary real analysis using tagged partitions. In this paper, we provide a set of alternative proofs based on the dyadic partitions. An important difference between tagged and dyadic partitions is that the results based on the dyadic partition can be obtained constructively, i.e. an algorithm is supplied to reach the conclusion of each proof. In addition, the construction involves only concepts and results presented in a first course real analysis, which is the reason why, comparing tagged partition and its theory, the proofs based on dyadic partition are more straightforward and accessible to a wide range of audience.

Keywords Dyadic interval, Real Analysis.
AMS subject classifications: 26A06, 26A42,

Similar as Gordon in [3], the purpose of this paper is to present alternate proofs of several well known results in elementary real analysis. The difference is that in this paper all the proofs are based on the dyadic partition and involves only the concepts and results in a first course real analysis which makes the proofs and reasonings more accessible to a wide range of audience. Because dyadic points follows a simple and precise generating pattern, proofs in this paper are often obtained constructively, i.e. an algorithm is supplied to reach the conclusion of each proof. Also, aside from results outlined in Gordon, this paper also proves Cauchy Mean Value Theorem, Gauss' Mean Value Theorem, as well as Darboux Mean Value Theorem. The directness of each proof, in our opinion, offers new insights to the fundamental theorems.

The results to be considered here all depend on the Completeness Axiom that every nonempty bounded set of real numbers has a supremum and its four equivalent results: 1. Every Cauchy sequence converges. 2. Every bounded monotone sequence converges. 3. Every bounded sequence contains a convergent subsequence. 4. The intersection of a nested sequence of closed and bounded intervals is nonempty.

For integers $j \geq 0$ and $0 \leq i \leq 2^j - 1$, we denote $x_{j,i} = a + \frac{i(b-a)}{2^j}$ and $I_{j,i} = [x_{j,i},\ x_{j,i+1}]$, the dyadic points and intervals of interval $[a, b]$, respectively (Figure 1). We also denote by $I^o_{j,i} = (x_{j,i},\ x_{j,i+1})$ and $D = \{x_{j,i};\ j \geq 0 \text{ and } 0 \leq i \leq 2^j - 1\}$, the set of all dyadic points of $[a, b]$. Since for any $x \notin D$, for each j, there exists a unique dyadic interval $I_{j,i}$, denote by $I_{j,x(j)}$, such that $x \in I_{j,x(j)}$ (note that $I_{0,x(0)} = [a, b]$). Thus, we have the following theorem.

Theorem 1 *If $x = a$, or $x = b$, or $x \notin D$, then there exists a unique nested sequence of dyadic intervals $\{I_{j,x(j)}\}$, i.e., $I_{j+1,x(j+1)} \subseteq I_{j,x(j)}$, such that $\{x\} = \bigcap_{j=0}^{\infty} I_{j,x(j)}$. For others $x \in D$, there are two nested sequences of dyadic intervals $\{I^r_{j,x(j)}\}$ and $\{I^l_{j,x(j)}\}$ such that $\{x\} = \bigcap_{j=0}^{\infty} I^r_{j,x(j)}$ and $\{x\} = \bigcap_{j=0}^{\infty} I^l_{j,x(j)}$, where $I^r_{j,x(j)}/I^l_{j,x(j)}$ means the dyadic interval taking x as the right/left end point if x is an end point of $I^r_{j,x(j)}/I^l_{j,x(j)}$ (Figure 1).*

For a function f defined on interval $[a, b]$, we denote

$$M_{j,i} = \sup\{f(x) :\ x \in I_{j,i}\}$$

if it is bounded above on $I_{j,i}$ and $M_{j,i} = \infty$ otherwise,

$$m_{j,i} = \inf\{f(x) :\ x \in I_{j,i}\}$$