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Minimizing movements for mean curvature flow of droplets with prescribed contact angle

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A B S T R A C T

We study the mean curvature motion of a droplet flowing by mean curvature on a horizontal hyperplane with a possibly nonconstant prescribed contact angle. Using the solutions constructed as a limit of an approximation algorithm of Almgren–Taylor–Wang and Luckhaus–Sturzenhecker, we show the existence of a weak evolution, and its compatibility with a distributional solution. We also prove various comparison results.

R É S U M É

Nous étudions le mouvement par courbure moyenne d'une goutte qui glisse par courbure moyenne sur un hyperplan horizontal avec un angle de contact prescrit éventuellement non constant. En utilisant les solutions construites comme limites d'un algorithme d'approximation dû à Almgren, Taylor et Wang et Luckhaus et Sturzenhecker, nous montrons l'existence d'une évolution faible, et sa compatibilité avec une solution au sens des distributions. Nous démontrons également plusieurs résultats de comparaison.

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1. Introduction

Historically, capillarity problems attracted attention because of their applications in physics, for instance in the study of wetting phenomena [18,22], energy minimizing drops and their adhesion properties [1,17,20,48], as well as because of their connections with minimal surfaces, see e.g. [14,29] and references therein.

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In this paper we are interested in the study of the evolution of a droplet flowing on a horizontal hyperplane under curvature driven forces with a prescribed (possibly nonconstant) contact angle. Although there are results in the literature describing the static and dynamic behaviours of droplets [2,12,49], not too much seems to be known concerning their mean curvature motion. Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [35,46,47,52] and zero-Neumann boundary condition [3,34,38,51]. It is also worthwhile to recall that, when the contact angle is constant, the evolution is related to the so-called mean curvature flow of surface clusters, also called space partitions (networks, in the plane): in two dimensions local well-posedness has been shown in [16], and authors of [39] derived global existence of the motion of grain boundaries close to an equilibrium configuration. See also [43] for related results. In higher space dimensions short time existence for symmetric partitions of space into three phases with graph-type interfaces has been derived in [30,31]. Very recently, authors of [26] have shown short time existence of the mean curvature flow of three surface clusters.

If we describe the evolving droplet by a set $E(t) \subset \Omega$, where $t \geq 0$ is the time, $\Omega = \mathbb{R}^n \times (0, +\infty)$ is the upper half-space in $\mathbb{R}^{n+1}$, the evolution problem we are interested in reads as

$$V = H_{E(t)} \qquad \text{on } \Omega \cap \partial E(t), \tag{1.1}$$

where V is the normal velocity and $H_{E(t)}$ is the mean curvature of $\partial E(t)$, supplied with the contact angle condition on the contact set (the boundary of the wetted area):

$$\nu_{E(t)} \cdot e_{n+1} = \beta \qquad \text{on } \overline{\Omega \cap \partial E(t)} \cap \partial\Omega, \tag{1.2}$$

where $\nu_{E(t)}$ is the outer unit normal to $\overline{\Omega \cap \partial E(t)}$ at $\partial\Omega$, and $\beta : \partial\Omega \to [-1, 1]$ is the cosine of the prescribed contact angle. We do not allow $\partial E(t)$ to be tangent to $\partial\Omega$, i.e. we suppose $|\beta| \leq 1 - 2\kappa$ on $\partial\Omega$ for some $\kappa \in (0, \frac{1}{2}]$. Following [38], in Appendix B we show local well-posedness of (1.1)–(1.2).

Short time existence describes the motion only up to the first singularity time. In order to continue the flow through singularities one needs a notion of weak solution. Concerning the case without boundary, there are various notions of generalized solutions, such as Brakke's varifold-solution [15], the viscosity solution (see [32] and references therein), the Almgren–Taylor–Wang [4] and Luckhaus–Sturzenhecker [41] solution, the minimal barrier solution (see [10] and references therein); see also [27,37] for other different approaches.

In the present paper we want to adapt the scheme proposed in [4,41], and later extended to the notions of *minimizing movement* and *generalized minimizing movement* by De Giorgi [25] (see also [6,8]), to solve (1.1)–(1.2). Let us recall the definition.

Definition 1.1. Let S be a topological space, $F : S \times S \times [1, +\infty) \times \mathbb{Z} \to [-\infty, +\infty]$ be a functional and $u : [0, +\infty) \to S$. We say that u is a generalized minimizing movement associated to F, S (shortly GMM) starting from $a \in S$ and we write $u \in GMM(F, S, \mathbb{Z}, a)$, if there exist $w : [1, +\infty) \times \mathbb{Z} \to S$ and a diverging sequence $\{\lambda_j\}$ such that

$$\lim_{j \to +\infty} w(\lambda_j, [\lambda_j t]) = u(t) \quad \text{for any } t \geq 0,$$

and the functions $w(\lambda, k)$, $\lambda \geq 1$, $k \in \mathbb{Z}$, are defined inductively as $w(\lambda, k) = a$ for $k \leq 0$ and

$$F(w(\lambda, k+1), w(\lambda, k), \lambda, k) = \min_{s \in S} F(s, w(\lambda, k), \lambda, k) \qquad \forall k \geq 0.$$

If $GMM(F, S, \mathbb{Z}, a)$ consists of a unique element we call it, for simplicity, a minimizing movement[1] starting from a.

[1] This slightly differs from the original definition in [25] which assumes the existence of $\lim_{\lambda \to +\infty} w(\lambda, [\lambda t]) = u(t)$.

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