J. Math. Pures Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

On an anisotropic Serrin criterion for weak solutions of the Navier–Stokes equations

Sur un critère de Serrin anisotrope pour des solutions faibles de l'équation de Navier–Stokes

Guillaume Lévy

Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Office 15-16 301, Paris, France

ARTICLE INFO

Article history:
Received 9 February 2017
Available online xxxx

MSC:
35B65
35Q30
35D30

Keywords:
Navier–Stokes equations
Weak solutions
Serrin theorem

ABSTRACT

In this paper, we draw on the ideas of [5] to extend the standard Serrin criterion [18] to an anisotropic version thereof. Because we work on weak solutions instead of strong ones, the functions involved have low regularity. Our method summarizes in a joint use of a uniqueness lemma in low regularity and the existence of stronger solutions. The uniqueness part uses duality in a way quite similar to the DiPerna–Lions theory, first developed in [7]. The existence part relies on L^p energy estimates, whose proof may be found in [5], along with an approximation procedure.

RÉSUMÉ

Dans cet article, nous utilisons les idées de [5] pour étendre le critère usuel de Serrin [18] à un cas anisotrope. Puisque nous nous intéressons à des solutiosn faibles et non fortes des équations de Navier–Stokes, les fonctions en jeu possèdent une faible régularité. Notre méthode peut se résumer à l'usage conjoint d'un lemme d'unicité en basse régularité avec un lemme d'existence de solutions plus régulières. La partie concernant l'unicité fait usage de la dualité d'une manière rappelant la théorie de Diperna–Lions, exposée pour la première fois dans [7]. La partie concernant l'existence repose sur des estimations d'énergie dans L^p, dont la preuve se trouve dans [5], ainsi que sur une procédure d'approximation standard.

1. Presentation of the problem

The present paper deals with the regularity of the Leray solutions of the incompressible Navier–Stokes equations in dimension three in space. We recall that these equations are

E-mail address: levy@ljll.math.upmc.fr.

https://doi.org/10.1016/j.matpur.2018.06.004

Please cite this article in press as: G. Lévy, On an anisotropic Serrin criterion for weak solutions of the Navier–Stokes equations, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.06.004

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p, \qquad t \geq 0,\ x \in \mathbb{X}^3, \\ \text{div } u \equiv 0, \\ u(0) = u_0. \end{cases} \tag{1}$$

Here, $u = (u^1, u^2, u^3)$ stands for the velocity field of the fluid, p is the pressure and we have set for simplicity the viscosity equal to 1. We use the letter $\mathbb{X}$ to denote $\mathbb{R}$ and $\mathbb{T}$ whenever the current claim or proposition applies to both of them. Let us first recall the existence theorem proved by J. Leray in his celebrated paper [13].

Theorem 1 *(J. Leray, 1934). Let us assume that u_0 belongs to the energy space $L^2(\mathbb{X}^3)$. Then there exists at least one vector field u in the energy space $L^\infty(\mathbb{R}_+, L^2(\mathbb{X}^3)) \cap L^2(\mathbb{R}_+, H^1(\mathbb{X}^3))$ which solves the system (1) in the weak sense. Moreover, the solution u satisfies for all $t \geq 0$ the energy inequality*

$$\frac{1}{2}\|u(t)\|^2_{L^2(\mathbb{X}^3)} + \int_0^t \|\nabla u(s)\|^2_{L^2(\mathbb{X}^3)} ds \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{X}^3)}.$$

Uniqueness of such solutions, however, remains an outstanding open problem to this day. In his paper from 1961 [18], J. Serrin proved that, if one assumes that there exists a weak solution which is mildly regular, then it is actually smooth in space. More precisely, J. Serrin proved that if a weak solution u belongs to $L^p(]T_1, T_2[, L^q(D))$ for $T_2 > T_1 > 0$ and some bounded domain $D \Subset \mathbb{X}$ with the restriction $\frac{2}{p} + \frac{3}{q} < 1$, then this weak solution is $\mathcal{C}^\infty$ in the space variable on $]T_1, T_2[\times D$. Following his path, many other authors proved results in the same spirit, with different regularity assumptions and/or covering limit cases. Let us cite for instance [3], [4], [5], [8], [9], [10], [11], [19], [20] and references therein. Closer to our paper is the result of J. Neustupa and P. Penel in [17]; their paper is the first about one-component regularity of weak solutions of the Navier–Stokes equations, though its main assumption is not scaling invariant.

In this paper, we prove two results of the type we mentioned above: the first one is stated in the torus, while the second one is in a spatial domain in the usual Euclidean space. Thanks to the compactness of the torus, the first result is easier to prove than its local-in-space counterpart. For this reason, we will use the torus case as a toy model, thus avoiding many technicalities and enlightening the overall strategy of the proof.

In the torus, the theorem writes as follows.

Theorem 2. *Let u be a Leray solution of the Navier–Stokes equations set in $\mathbb{R}_+ \times \mathbb{T}^3$*

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u) - \Delta u = -\nabla p \\ u(0) = u_0 \end{cases}$$

with initial data u_0 in $L^2(\mathbb{T}^3)$ and assume that there exists a time interval $]T_1, T_2[$ such that its third component u^3 satisfies

$$u^3 \in L^2(]T_1, T_2[, W^{2,\frac{3}{2}}(\mathbb{T}^3)).$$

Then u is actually smooth in space on $]T_1, T_2[\times \mathbb{T}^3$.

In a subdomain of the whole space, we need to add a technical assumption on the initial data, namely that it belongs to some particular L^p space with $p < 2$. Notice that such an assumption is automatically satisfied in the torus, thank to its compactness.

Please cite this article in press as: G. Lévy, On an anisotropic Serrin criterion for weak solutions of the Navier–Stokes equations, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.06.004