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Uniqueness and regularity of conservative solution to a wave system modeling nematic liquid crystal

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A B S T R A C T

In this paper, we prove the uniqueness and generic regularity of the energy conservative solution for a system of wave equations modeling nematic liquid crystal.

R É S U M É

Dans cet article, nous prouvons l'unicité et la régularité générique de la solution de conservation d'énergie pour un système d'équations d'onde modélisant le cristal liquide nématique.

1. Introduction

In this paper, we study the uniqueness and generic regularity for energy conservative Hölder continuous solution to the system of wave equations

$$\partial_{tt} n_i - \partial_x (c^2(n_1)\partial_x n_i) = \big(-|\mathbf{n}_t|^2 + (2c^2 - \lambda_i)|\mathbf{n}_x|^2\big) n_i, \qquad i = 1, 2, 3, \tag{1.1}$$

on $\mathbf{n} = (n_1, n_2, n_3)$ with

$$|\mathbf{n}| = 1. \tag{1.2}$$

Here, the time t and space variables x belong to $\mathbb{R}^+$ and $\mathbb{R}$, respectively. The constants

$$\lambda_1 = \gamma > 0 \quad \text{and} \quad \lambda_2 = \lambda_3 = \alpha > 0.$$

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The (positive) wave speed c depends on n_1 with

$$c^2(n_1) = \alpha + (\gamma - \alpha)n_1^2. \tag{1.3}$$

The initial data are

$$n_i|_{t=0} = n_{i0} \in H^1, \quad (n_i)_t|_{t=0} = n_{i1} \in L^2, \quad i = 1,2,3. \tag{1.4}$$

We briefly introduce the origin of system (1.1) from nematic liquid crystal. Liquid crystal is often viewed as an intermediate state between liquid and solid. It possesses none or partial positional order but displays an orientational order at the same time. For the nematic phase, the molecules float around as in a liquid phase, but have the tendency of aligning along a preferred direction due to their orientation. The mean orientation of the long molecules in a nematic liquid crystal is described by a director field of unit vectors, $\mathbf{n} \in \mathbb{S}^2$, the unit sphere. Associated with the director field $\mathbf{n}$, there is the well-known Oseen–Franck potential energy density W given by

$$W(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}\alpha(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}\beta\left(\mathbf{n}\cdot\nabla\times\mathbf{n}\right)^2 + \frac{1}{2}\gamma\left|\mathbf{n}\times(\nabla\times\mathbf{n})\right|^2. \tag{1.5}$$

The positive constants α, β, and γ are elastic constants of the liquid crystal, corresponding to splay, twist, and bend, respectively.

There are many studies on the constrained elliptic system of equations for $\mathbf{n}$ derived through variational principles from the potential (1.5), and on the parabolic flow associated with it, see [1–6] and references therein.

In the regime in which inertia effects dominate viscosity, the propagation of the orientation waves in the director field may then be modeled by the least action principle ([7,8])

$$\frac{\delta}{\delta\mathbf{n}}\int_{\mathbb{R}^+}\int_{\mathbb{R}^3}\left\{\frac{1}{2}\partial_t\mathbf{n}\cdot\partial_t\mathbf{n} - W(\mathbf{n},\nabla\mathbf{n})\right\}d\mathbf{x}\,dt = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1. \tag{1.6}$$

When the space dimension is one (1-d), i.e. $x \in \mathbb{R}$, and when $\alpha = \beta$, system (1.6) exactly gives (1.1), on which we focus in this paper. There is a simpler case when $\mathbf{n} = (\cos u(x,t), \sin u(x,t), 0)$ (planar deformation). In this case, the function u satisfies

$$u_{tt} - c(u)(c(u)\,u_x)_x = 0, \tag{1.7}$$

with $c^2(u) = \gamma\cos^2 u + \alpha\sin^2 u$. See [9] and [10] for the derivations of (1.1) and (1.7).

Because of the strong nonlinearity, the solution for (1.6) fails to be Lipschitz continuous even for 1-d solution with C^∞ initial data, such as for solutions of (1.1) and (1.7). See [11] for an example with finite time gradient blowup. More precisely, the 1-d solution in general includes cusp singularity, which means that solution is only Hölder continuous, due to the energy concentration. This causes the following major difficulties in studying the existence, uniqueness and Lipschitz continuous dependence of global weak solution respectively:

- Classical solution in general does not exist. One has to study weak solutions.
- Solution in general is not unique. To select a unique solution, one needs a physical admissible condition, such as the energy conservation used in this paper. However, the energy conservation laws are only in the weak form.

  Another type of solutions are called dissipative solutions. See existence of dissipative solution for (1.7) with monotonic wave speed $c(\cdot)$ in [12–14].

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