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Rank-one convexity implies polyconvexity in isotropic planar incompressible elasticity

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We study convexity properties of energy functions in plane nonlinear elasticity of incompressible materials and show that rank-one convexity of an objective and isotropic elastic energy W on the special linear group $\mathrm{SL}(2)$ implies the polyconvexity of W.

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R É S U M É

Nous étudions les propriétés de convexité des fonctions énergie pour des matériaux incompressibles dans le cas de l'élasticité non-linéaire plane. Nous montrons que la convexité de rang 1 d'une fonction énergie isotrope et élastique W dans le groupe spécial linéaire $\mathrm{SL}(2)$ implique la polyconvexité de W.

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1. Introduction

The aim of this paper is to study the relation between rank-one convexity and polyconvexity of *objective* and *isotropic* real valued functions W on $\mathrm{SL}(2) = \{X \in \mathbb{R}^{2\times 2} \mid \det X = 1\}$. These convexity properties play an important role in the theory of nonlinear hyperelasticity, where $W(\nabla\varphi)$ is interpreted as the energy density of a deformation $\varphi\colon \Omega \to \mathbb{R}^2$; here, $\Omega \subset \mathbb{R}^2$ corresponds to a planar elastic body in its reference configuration. In particular, energy functions on the domain $\mathrm{SL}(2)$ are used for mod-

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elling *incompressible* materials, since in this case, the deformation φ is subject to the additional constraint $\det \nabla\varphi = 1$.[1]

The notion of polyconvexity was introduced into the context of nonlinear elasticity theory by John Ball [5,6] (cf. [34,11,39]). Polyconvexity criteria in the case of spatial dimension 2 were conclusively discussed by Rosakis [36] and Šilhavý [41–44,46–48], while an exhaustive self-contained study giving necessary and sufficient conditions for polyconvexity in arbitrary spatial dimension was given by Mielke [22]. Rank-one convexity plays an important role in the existence and uniqueness theory for linear elastostatics and elastodynamics [30,16,14,15,19]. Criteria for the rank-one convexity of functions defined on $\mathrm{GL}^+(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X > 0\}$ were established by Knowles and Sternberg [18] as well as by Šilhavý [43,45], Dacorogna [10], Aubert [4] and Davies [12].

It is well known that the implications

$$\text{polyconvexity} \quad\Longrightarrow\quad \text{quasiconvexity} \quad\Longrightarrow\quad \text{rank-one convexity}$$

hold for functions on $\mathbb{R}^{n\times n}$ (as well as for functions on $\mathrm{SL}(n)$, see [9, Theorem 1.1]) for arbitrary dimension n. The reverse implications, on the other hand, do not hold in general: rank-one convexity does not imply polyconvexity [2] for dimension $n \geq 2$, and rank-one convexity does not imply quasiconvexity [7,49,33,11] for $n > 2$. Whether this latter implication holds for $n = 2$ is still an open question: the conjecture that rank-one convexity and quasiconvexity are *not* equivalent for $n = 2$ is also called *Morrey's conjecture* [24]. For certain classes of functions on $\mathbb{R}^{2\times 2}$, however, it has been demonstrated that the two convexity properties are, in fact, equivalent [50,40,20,49,25,8,7,32,31,27].

In a previous paper [21], we have shown that any energy function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ which is isotropic and objective (i.e. bi-SO(2)-invariant) as well as *isochoric*[2] is rank-one convex if and only if it is polyconvex. In January 2016, a question by John Ball motivated some investigation into whether this result might be applicable to the incompressible case. In March 2016, at the Joint DMV and GAMM Annual Meeting in Braunschweig, Alexander Mielke indicated that some of his results [22] should be suitable for this task.

The main result of the present paper is Theorem 3.1, which states that for objective and isotropic energies on $\mathrm{SL}(2)$, rank-one convexity implies (and is therefore equivalent to) polyconvexity. Theorem 3.1 includes a slightly stronger two-dimensional version of a criterion by Dunn, Fosdick and Zhang (cf. Section 2.2): an energy W with $W(F) = \phi(\sqrt{\|F\| - 2})$ for $F \in \mathrm{SL}(2)$ is polyconvex on $\mathrm{SL}(2)$ (if and only if it is rank one convex) if and only if ϕ is nondecreasing and convex, regardless of any regularity assumption on the energy.[3]

2. Rank-one convexity and polyconvexity on SL(2)

We consider the concepts of *rank-one convexity* and *polyconvexity* of real-valued objective, isotropic functions W on the group $\mathrm{GL}^+(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X > 0\}$ and on its subgroup $\mathrm{SL}(2) = \{X \in \mathbb{R}^{2\times 2} \,|\, \det X = 1\}$. We denote by λ_1, λ_2 the singular values of F (i.e. the eigenvalues of $U = \sqrt{F^T F}$), and $\lambda_{\max} := \max\{\lambda_1, \lambda_2\}$ denotes the largest singular value of F (also called the *spectral norm* of F). The elastic energy W is assumed to be *objective* as well as *isotropic*, i.e. to satisfy the equality

[1] Note that a function W defined only on $\mathrm{SL}(2)$ can equivalently be expressed as a (discontinuous) function $W\colon \mathbb{R}^{2\times 2} \to \mathbb{R}\cup\{+\infty\}$ with $W(F) = +\infty$ for all $F \notin \mathrm{SL}(2)$. This interpretation of functions not defined on all of $\mathbb{R}^{2\times 2}$ is reflected by Mielke's definition of polyconvexity [22] of energies W on $\mathrm{SL}(2)$, see Definition 2.2.

[2] A function $W\colon \mathrm{GL}^+(2) \to \mathbb{R}$ is called isochoric if $W(a\,F) = W(F)$ for all $a \in (0,\infty)$. Some relations between isotropic, objective and isochoric energies and the functions defined on $\mathrm{SL}(2)$ are discussed in Section 4. In elasticity theory, isochoric energy functions measure only the *change of form* of an elastic body, not the *change of size*.

[3] Throughout this article, $\|X\|^2 = \langle X, X\rangle$ denotes the Frobenius tensor norm of $X \in \mathbb{R}^{n\times n}$, where $\langle X, Y\rangle = \mathrm{tr}(Y^T X)$ is the standard Euclidean scalar product on $\mathbb{R}^{n\times n}$. The identity tensor on $\mathbb{R}^{n\times n}$ will be denoted by $\mathbb{1}$, so that $\mathrm{tr}\,(X) = \langle X, \mathbb{1}\rangle$.

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