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Spectra of Bernoulli convolutions and random convolutions [☆]

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ABSTRACT

In this work we study the harmonic analysis of infinite convolutions generated by compatible pairs. We first give some sufficient conditions so that a random infinite convolution μ becomes a spectral measure, i.e., there exists a countable set $\Lambda \subseteq \mathbb{R}^n$ such that $E(\Lambda) = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis for $L^2(\mu)$. As applications, we settle down the spectral eigenvalue problem for spectral Bernoulli convolutions.

RÉSUMÉ

Dans ce travail nous étudions l'analyse harmonique de convolutions infinies engendrées par des paires compatibles. Nous donnons d'abord des conditions suffisantes telles qu'une convolution infinie aléatoire μ devient une mesure spectrale, i.e., il existe un ensemble dénombrable $\Lambda \subseteq \mathbb{R}^n$ tel que $E(\Lambda) = \{e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ offre une base orthonormée pour $L^2(\mu)$. Comme application, nous tranchons le problème de valeurs propres spectrales pour les convolutions spectrales de Bernoulli.

1. Introduction

For a compactly supported Borel probability measure μ on $\mathbb{R}^n$, we call μ a *spectral measure* if there exists a countable set $\Lambda \subseteq \mathbb{R}^n$ such that the family of complex exponentials $E(\Lambda) := \{e_\lambda(x) = e^{2\pi i\langle\lambda,x\rangle} : \lambda \in \Lambda\}$ forms an orthonormal basis (Fourier basis) for $L^2(\mu)$. In this case, the set Λ is called a *spectrum* for μ, and we also say that (μ, Λ) forms a *spectral pair*. When μ is the normalized Lebesgue measure supported on a

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Borel set Ω, the existence of a spectrum is closely related to the famous Fuglede conjecture, which asserts that there exists a spectrum for μ if and only if Ω is a translational tile [21]. This conjecture has been proved to be false by Tao and others in both directions in dimension three or higher, see, e.g., [30,40], but it is still open in dimension 1 and 2.

The first non-atomic, singular continuous spectral measure was found by Jorgensen and Pedersen [28] in 1998, which opened up a new field in researching the orthogonal harmonic analysis of fractal measures including self-similar/self-affine measures and generally Moran measures, see [35,36,29,38]. Nowadays, there has been a wide range of interests in the study of harmonic analysis on fractals. Many interesting spectral measures have been found, at the same time some singular phenomena different from the spectral theory of Lebesgue measures have been discovered, please see [1–3,5,7,9,10,13,8,19,14–16,39,26,25,6,22] and the references therein. In all these research, the following two types of problems are basic for the spectral measure theory:

I. **Spectral Problem**: For what measure μ does $L^2(\mu)$ admit a Fourier basis?

II. **Spectral Eigenvalue Problem**: This is a particular problem for singular spectral measures. There are two basic forms: (1) Let Λ be a spectrum for a spectral measure μ. Find all real numbers p such that $p\Lambda$ is also a spectrum for μ; (2) Let μ be a spectral measure. Find all real numbers p for which there exists a set Λ such that both Λ and $p\Lambda$ are spectra for μ. In the two cases, p is called a *spectral eigenvalue* of μ and Λ is called an *eigen-spectrum* of μ corresponding to p.

In this paper, we will deal with the above two problems for some infinite convolutions generated by finite discrete measures, in particular, for Bernoulli convolutions $\mu_{2k}, k \in \mathbb{N}$. Recall that the Bernoulli convolution μ_R for each $R > 1$ is the distribution of the random variable $\sum_{n=1}^{\infty} \pm R^{-n}$, where the signs "+" and "−" are chosen independently with probability 1/2. Bernoulli convolutions have been studied extensively in many areas of mathematics including Fourier analysis, dynamical system, integer tile, wavelet theory, algebraic number theory and fractal geometry since 1930s (e.g., see [41] and the references therein) and have the following expression of convolutions:

$$\mu_R := \mu_{R,D} = \delta_{R^{-1}D} * \delta_{R^{-2}D} * \cdots * \delta_{R^{-n}D} * \cdots, \quad \text{where} \quad D = \{-1, 1\}. \tag{1.1}$$

Here, the symbol δ_E for a finite set E denotes the atomic measure

$$\delta_E = \frac{1}{\#E} \sum_{e \in E} \delta_e,$$

where δ_e is the Dirac point mass measure at the point e, $rE = \{re : e \in E\}$ and $\#E$ is the cardinality of E.

Moreover, the measure μ_R is the unique probability one with compact support satisfying that

$$\mu_R(\cdot) = \frac{1}{2}\mu_R \circ \tau_-^{-1}(\cdot) + \frac{1}{2}\mu_R \circ \tau_+^{-1}(\cdot),$$

where $\{\tau_+(x) = R^{-1}(x+1),\ \tau_-(x) = R^{-1}(x-1)\}$ is an *iterated function system* (IFS) on $\mathbb{R}$ [24]. The study on the spectral property of Bernoulli convolutions dates back to the work of Jorgensen and Pedersen [28], and many spectral properties of Bernoulli convolutions have been found in [26,14,9,3,8,5] and references cited therein. An outstanding result of Jorgensen and Pedersen [28], Dai [5] shows that

Theorem A. *A Bernoulli convolution μ_R is a spectral measure if and only if R is a positive even integer. Moreover, let $C = \{0, \frac{k}{2}\}$ for integer $k \geq 1$, then the set*

$$\Lambda(2k, C) := \left\{ \sum_{j=0}^{m} (2k)^j c_j : \ c_j \in C, \ m \geq 0 \right\} \tag{1.2}$$

is a spectrum for μ_{2k}.