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# Correlation based passive imaging with a white noise source

T. Helin [a], M. Lassas [a], L. Oksanen [b], T. Saksala [a,*]

[a] Department of Mathematics and Statistics, University of Helsinki, Finland
[b] Department of Mathematics, University College London, United Kingdom

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ABSTRACT

Passive imaging refers to problems where waves generated by unknown sources are recorded and used to image the medium through which they travel. The sources are typically modelled as a random variable and it is assumed that some statistical information is available. In this paper we study the stochastic wave equation $\partial_t^2 u - \Delta_g u = \chi W$, where $W$ is a random variable with the white noise statistics on $\mathbb{R}^{1+n}$, $n \geq 3$, $\chi$ is a smooth function vanishing for negative times and outside a compact set in space, and $\Delta_g$ is the Laplace–Beltrami operator associated to a smooth non-trapping Riemannian metric tensor $g$ on $\mathbb{R}^n$. The metric tensor $g$ models the medium to be imaged, and we assume that it coincides with the Euclidean metric outside a compact set. We consider the empirical correlations on an open set $\mathcal{X} \subset \mathbb{R}^n$,

$$C_T(t_1, x_1, t_2, x_2) = \frac{1}{T}\int_0^T u(t_1 + s, x_1)u(t_2 + s, x_2)ds, \quad t_1, t_2 > 0, \ x_1, x_2 \in \mathcal{X},$$

for $T > 0$. Supposing that $\chi$ is non-zero on $\mathcal{X}$ and constant in time after $t > 1$, we show that in the limit $T \to \infty$, the data $C_T$ becomes statistically stable, that is, independent of the realization of $W$. Our main result is that, with probability one, this limit determines the Riemannian manifold $(\mathbb{R}^n, g)$ up to an isometry.


RÉSUMÉ

L'imagerie passive concerne des problémes oú des ondes générées par des sources inconnues sont enregistrées et ensuite utilisées pour reconstruire le milieu dans lequel elles se propagent. Les sources sont typiquement modélisées par des variables aléatoires dont la distribution statistique du bruit est connu, et on suppose de connaitre quelque information statistique. Dans cet article nous étudions l'équation stochastique des ondes $\partial_t^2 u - \Delta_g u = \chi W$, où $W$ est une variable aléatoire ayant pour distribution un bruit blanc sur $\mathbb{R}^{1+n}$, $n \geq 3$, $\chi$ est une fonction régulière qui s'annule pour des temps négatifs et en dehors d'un ensemble compact dans l'espace, et $\Delta_g$ est l'opérateur de Laplace–Beltrami associé à un tenseur métrique Riemannien régulier et *non-trapping* $g$ sur $\mathbb{R}^n$. Le tenseur métrique $g$ modélise le milieu que l'on veut imager, et nous supposons qu'il coincide avec la métrique Euclidienne en dehors

* Corresponding author.
*E-mail addresses:* tapio.helin@helsinki.fi (T. Helin), matti.lassas@helsinki.fi (M. Lassas), l.oksanen@ucl.ac.uk (L. Oksanen), teemu.saksala@helsinki.fi (T. Saksala).

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d'un ensemble compact. Nous considérons les correlations empiriques sur un ouvert $\mathcal{X} \subset \mathbb{R}^n$,

$$C_T(t_1, x_1, t_2, x_2) = \frac{1}{T} \int_0^T u(t_1 + s, x_1) u(t_2 + s, x_2) ds, \quad t_1, t_2 > 0, \; x_1, x_2 \in \mathcal{X},$$

pour $T > 0$. Sous l'hypothèse que $\chi$ ne s'annule pas entièrement sur $\mathcal{X}$ et reste constante dans le temps pour $t > 1$, nous montrons que la donnée $C_T$, lorsque $T \to \infty$, devient statistiquement stable, c'est-à-dire indépendante de la réalisation de $W$. Notre résultat principal est que cette limite, en probabilité, détermine la variété Riemannienne $(\mathbb{R}^n, g)$ à une isométrie près.



## 1. Introduction

In passive imaging, waves generated by unknown sources are recorded and used to image the medium through which they travel. Passiveness refers to the observer having only little or no control over the source (consider earthquakes in seismic imaging). However, some statistical information of the source may be available and it can be useful to model the source as a random variable: while the statistics of the random variable is known, its realization remains unknown.

Passive imaging has had a fundamental impact to seismic and various other imaging modalities. We refer to the recent monograph by Garnier and Papanicolaou [25] for an extensive review of the field. The present paper is in the spirit of Chapter 2 of [25] where the asymptotic parameter is the correlation integration time, that is, $T$ in equation (2) below.

We will compare below our result with the previous results of similar nature, but the strength of the present result lies in the geometric generality involved. The exact recovery of an unknown medium is possible with spatially limited noise source distribution and geometric assumptions that are independent of the spatial support of the noise source. The geometric assumptions are made only to guarantee that local energy decay holds for the wave equation considered, this decay playing a similar role as the constant damping in previous literature, see [18,3] and [25, Prop. 2.4].

We consider the wave equation

$$\begin{aligned} &\partial_t^2 u(t,x) - \Delta_g u(t,x) = \chi(t,x) W(t,x) \quad \text{in } \mathbb{R}_+^{1+n} = (0,\infty) \times \mathbb{R}^n, \\ &u|_{t=0} = \partial_t u|_{t=0} = 0, \end{aligned} \tag{1}$$

where $\Delta_g$ is the Laplace–Beltrami operator corresponding to a smooth time-independent Riemannian metric $g$ on $\mathbb{R}^n$. Here the vanishing initial conditions in (1) are interpreted in the sense that $u$ is supported in $[0,\infty) \times \mathbb{R}^n$. In coordinates $(x_j)_{j=1}^n$ the Laplace–Beltrami operator has the following representation

$$\Delta_g = \sum_{j,k=1}^n |g|^{-1/2} \frac{\partial}{\partial x^j} \left( |g|^{1/2} g^{jk} \frac{\partial}{\partial x^k} u \right),$$

where $[g_{jk}]_{j,k=1}^n = g(x)$, $|g| = \det(g_{jk})$ and $[g^{jk}]_{j,k=1}^n = g(x)^{-1}$.

We assume that our source $W$ is a realization of a Gaussian white noise random variable on $\mathbb{R}^{1+n}$. Moreover, $\chi$ stands for a smooth function

$$\chi(t,x) = \chi_0(t)\kappa(x),$$

such that $\chi_0 \in C^\infty(\mathbb{R})$ and

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