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Regularity of intrinsically convex $W^{2,2}$ surfaces and a derivation of a homogenized bending theory of convex shells

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ABSTRACT

We prove interior regularity for $W^{2,2}$ isometric immersions of surfaces endowed with a smooth Riemannian metric of positive Gauss curvature.

We then derive the Γ-limit of three dimensional nonlinear shells with inhomogeneous energy density, in the bending energy regime. This derivation is incomplete in that it requires an additional technical hypothesis.

RÉSUMÉ

Nous prouvons un résultat de régularité intérieure pour des immersions isometriques $W^{2,2}$ de surfaces munies d'une metrique riemannienne reguliere de courbure de Gauss positive. Nous derivons la Γ-limite de coques en flexion non linéaires et non homogènes. Ce résultat est incomplet puisqu'il nécessite une hypothèse (technique) additionnelle.

1. Introduction

For C^2 isometric immersions u of a two-dimensional Riemannian manifold with positive Gauss curvature into $\mathbb{R}^3$, there is a link between the regularity of the metric and the regularity of u; in particular, if the metric is smooth then so is u. Without a priori assumptions on the regularity of u this link is broken.

In the present paper, we show that square integrability of the second fundamental form of u is sufficient for the link to persist. In particular, if the metric is smooth, then u is smooth in the interior, provided that initially it belongs to the Sobolev space $W^{2,2}$.

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Our regularity results for metrics with positive Gauss curvature rely upon earlier work by Šverák on the Monge–Ampère equation. Due to the low regularity, the passage from the scalar problem to the vectorial problem addressed here is not trivial.

Relaxing C^2 regularity to regularity on the Sobolev scale is important for variational problems: the $W^{2,2}$ isometric immersions studied here arise naturally in thin film elasticity. In the present paper, we use this regularity result to derive homogenized bending models for convex shells from three dimensional nonlinear elasticity.

Regarding shells theories in elasticity, we refer to [8] for an overview of the derivation via formal asymptotic expansions. In the case of linearly elastic shells, these models can also be justified rigorously.

More recently, nonlinear models for rods, curved rods, plates and shells have been derived rigorously by means of Γ-convergence, starting from three dimensional nonlinear elasticity. The first results in that direction can be found in [1,18,19]. The nonlinear bending theory for plates was derived in [10], and the corresponding theory for shells in [9].

In the second part of this article we derive a homogenized nonlinear bending theory of shells, by simultaneous homogenization and dimension reduction. This generalizes the results from [9]. Our starting point is the energy functional of three dimensional nonlinear elasticity: We consider a reference configuration which is a shell $S^h \subset \mathbb{R}^3$ of thickness $h > 0$ around an embedded surface $S \subset \mathbb{R}^3$. The elastic energy stored in the deformed configuration determined by a deformation $u \in W^{1,2}(S^h, \mathbb{R}^3)$ is given by

$$\frac{1}{h^2\,|S^h|}\int_{S^h} W_\varepsilon(x, \nabla u(x))\,dx. \tag{1}$$

The function W_ε is a stored energy function that oscillates periodically in x, with some period $\varepsilon \ll 1$. We are interested in the effective behavior of the functionals (1) when both the thickness h and the period ε are small: we consider the asymptotic behavior of (1) when h and ε tend to zero simultaneously.

Such a combination of dimension reduction and homogenization was studied, e.g., in [4]. More recently, homogenized nonlinear plate theories in the von Kármán energy regime and in the bending regime were studied in [24] and in [15,31]. In these cases one does not obtain an infinite-cell homogenization formula as in the membrane case studied in [4]. This is because for small strains the energy is essentially convex, so one can use two-scale convergence techniques.

The derivation of a homogenized theory of shells in the von Kármán energy regime was carried out in [16]. Different models were obtained in the regime $h \ll \varepsilon$. For generic shells, the models for the situations $\varepsilon^2 \lesssim h \ll \varepsilon$ have been derived. For convex shells, the whole regime $h \ll \varepsilon$ is now understood.

The geometric framework developed in [16] will be used in the present paper as well. Here we are interested in the analogous theory for the bending energy regime. We restrict ourselves to convex shells. Our main result in this direction is Theorem 3.2. The derivation of the lower bound is quite natural. However, as usual, we can prove sharpness of the lower bound only for regular limiting deformations. We are not able to close this regularity gap. However, our regularity result Theorem 2.1 allows us to narrow the gap: using it, we can construct the required recovery sequence starting from a limiting deformation which is not in $W^{3,\infty}$, but merely in $W^{2,\infty}$. In addition, Theorem 2.1 confirms the intuition that all finite energy deformations of a convex shell preserve convexity.

2. Regularity of intrinsically convex $W^{2,2}$ surfaces

The purpose of this chapter is to prove the following result:

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