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The 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion

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A B S T R A C T

This paper examines the global regularity problem on the two-dimensional (2D) incompressible Boussinesq equations with fractional horizontal dissipation and thermal diffusion. The goal is to establish the global existence and regularity for the Boussinesq equations with minimal dissipation and thermal diffusion. By working with this general 1D fractional Laplacian dissipation, we are no longer constrained to the standard partial dissipation and this study will help understand the issue on how much dissipation is necessary for the global regularity. Due to the nonlocality of these 1D fractional operators, some of the standard energy estimate techniques such as integration by parts no longer apply and new tools including several anisotropic embedding and interpolation inequalities involving fractional derivatives are derived. These tools allow us to obtain very sharp upper bounds for the nonlinearities.

R É S U M É

Cet article examine le problème de la régularité globale sur les équations de Boussinesq bi-dimensionnels (2D) incompressibles avec dissipation horizontale fractionnaire et avec diffusion thermique. L'objectif est d'établir l'existence globale et la régularité pour les équations de Boussinesq avec dissipation minimale et diffusion thermique. En travaillant avec cette dissipation laplacienne fractionnaire unidimensionnelle assez générale, nous ne sommes plus limités à la dissipation partielle standard, et cette etude nous aidera à comprendre le problème sur combien de dissipation est nécessaire pour obtenir la régularité globale. A cause de la non-localité de ces opérateurs fractionnaires unidimensionnels, certaines des techniques d'estimation d'énergie standard, par exemple l'intégration par partie, ne s'applique plus, et des nouveaux outils comprenant l'injection anisotrope et les inégalités

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d'interpolation concernant des dérivés fractionnaires sont dérivées. Ces outils nous permettent d'obtenir des bornes supérieures très sharp pour les non-linéarités.

1. Introduction

This paper concerns itself with the initial-value problem for the 2D Boussinesq equations with fractional horizontal dissipation and thermal diffusion,

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\Lambda_{x_1}^{2\alpha}u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\Lambda_{x_1}^{2\beta}\theta = 0, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{1.1}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, $e_2 = (0,1)$, $u = (u_1(x,t), u_2(x,t))$ denotes the velocity field, $p = p(x,t)$ the pressure, $\theta = \theta(x,t)$ the temperature, and $\mu > 0$, $\nu > 0$, $\alpha \in [0,1]$ and $\beta \in [0,1]$ are real parameters. The horizontal fractional operator $\Lambda_{x_1} := \sqrt{-\partial_{x_1}^2}$ is defined through the Fourier transform, namely

$$\widehat{\Lambda_{x_1}^{\gamma} f}(\xi) = |\xi_1|^\gamma \hat{f}(\xi).$$

The goal here is to show the global regularity for (1.1) for smallest $\alpha, \beta \in [0,1]$.

We summarize some previous work closely related to our study here. To facilitate the description, we start with the general form of the 2D incompressible Boussinesq equations given by

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \mu\mathcal{L}_1 u + \nabla p = \theta e_2, \\ \partial_t\theta + (u\cdot\nabla)\theta + \nu\mathcal{L}_2\theta = 0, \\ \nabla\cdot u = 0, \end{cases} \tag{1.2}$$

where $\mathcal{L}_1$ and $\mathcal{L}_2$ are Fourier multiplier operators, namely

$$\widehat{\mathcal{L}_1 u}(\xi) = m_1(\xi)\,\widehat{u}(\xi), \qquad \widehat{\mathcal{L}_2 u}(\xi) = m_2(\xi)\,\widehat{u}(\xi).$$

When $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$, (1.2) becomes the standard model in geophysics as well as in the Rayleigh–Bérnard convection (see, e.g., [26,28,31]). When $\mu = \nu = 0$, (1.2) becomes completely inviscid. When $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by various special symbols, we recover various partial and fractional dissipation cases, which naturally bridges the fully dissipative Boussinesq and the complete inviscid Boussinesq equation.

The global regularity problem on (1.2) with partial or fractional dissipation has attracted considerable interests and there have been substantial developments. The global regularity for (1.2) with $\mathcal{L}_1 = \mathcal{L}_2 = -\Delta$ can be obtained via similar approaches as those for the 2D Navier–Stokes equations. In fact, any L^2-initial data (u_0, θ_0) leads to a unique global solution to the fully dissipative 2D Boussinesq equation that becomes infinitely smooth for any time $t > 0$. In contrast, the completely inviscid Boussinesq equation is not well-understood and the global well-posedness remains an outstanding open problem. Due to the similarity between the 2D inviscid Boussinesq and the 3D axisymmetric Euler equations, the finite time singularity indicated by the numerical simulations on the 3D Euler in a bounded domain with special geometry and boundary data exhibits the complexity of this problem [25]. Sarria and Wu examined a special class of singular solutions [29]. Two 1D models as well as several 2D models of the inviscid Boussinesq equations

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