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Compactness, existence and multiplicity for the singular mean field problem with sign-changing potentials

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ABSTRACT

In this paper we consider a mean field problem on a compact surface without boundary in presence of conical singularities. The corresponding equation, named after Liouville, appears in the Gaussian curvature prescription problem in Geometry, and also in the Electroweak Theory and in the abelian Chern–Simons–Higgs model in Physics. Our contribution focuses on the case of sign-changing potentials, and gives results on compactness, existence and multiplicity of solutions.

RÉSUMÉ

Dans cet article on considère un problème de champ moyen sur une surface compacte et sans bord, en présence de singularités coniques. L'équation correspondante, dite de Liouville, apparaît en Géométrie dans le problème de la courbure de Gauss prescrite et en Physique, soit dans le cadre de la Théorie Electrofaible, soit dans le modèle de Chern–Simons–Higgs abélien. Notre contribution porte sur le cas des potentiels changeant de signe et donne des résultats de compacité, d'existence et de multiplicité de solutions.

1. Introduction

The classical problem of prescribing the Gaussian curvature on a compact surface Σ under a conformal change of the metric dates back to [12,37]. Let us denote by g the original metric, $\tilde{g}$ the new one and e^v the conformal factor (that is, $\tilde{g} = e^v g$). The problem reduces to solving the PDE

$$-\Delta_g v + 2K_g(x) = 2K(x)e^v,$$

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where K_g, K denote the curvature with respect to g and $\tilde{g}$, respectively. Observe that by the Uniformization Theorem we can assume that K_g is a constant. The solvability of this equation has been studied for a long time, and it is not possible to give here a comprehensive list of references.

The above setting needs to be modified if one wants to prescribe also the appearance of conical singularities on the surface, a case that was first studied in [51]. We recall that a conformal metric $\tilde{g}$ has a conical singularity at p of order $\alpha \in (-1, +\infty)$ if there exist local coordinates z in $\mathbb{C}$ such that $z(p) = 0$ and

$$\hat{g}(z) = e^{\psi}|z|^{2\alpha}|dz|^2, \tag{1.1}$$

where $\hat{g}$ is the local expression of g and ψ is continuous in a neighborhood of 0 in $\mathbb{C}$ and C^2 outside the point p.

In this case we are led with weak solutions of the problem

$$-\Delta_g v + 2K_g = 2K(x)e^v - 4\pi \sum_{j=1}^{m} \alpha_j \delta_{p_j}, \tag{1.2}$$

where δ_{p_j} denotes a Dirac delta at the point $p_j \in \Sigma$ (see Appendix of [3] for a rigorous deduction of (1.2)). Integrating the above equation and taking into account the Gauss–Bonnet formula, we obtain

$$4\pi\chi(\Sigma) = 2\int_{\Sigma} Ke^v \, dV_g - 4\pi \sum_{j=1}^{m} \alpha_j. \tag{1.3}$$

We now transform equation (1.2) into another one which admits a variational structure. Let $G(x, y)$ be the Green function of the Laplace–Beltrami operator on Σ associated to g, i.e.

$$-\Delta_g G(x, y) = \delta_y - \frac{1}{|\Sigma|} \quad \text{in} \quad \Sigma, \qquad \int_{\Sigma} G(x, y) dV_g(x) = 0. \tag{1.4}$$

We define

$$h_m(x) = 4\pi \sum_{j=1}^{m} \alpha_j G(x, p_j) = 2\sum_{j=1}^{m} \alpha_j \log\left(\frac{1}{d(x, p_j)}\right) + 2\pi\alpha_j H(x, p_j), \tag{1.5}$$

where H is the regular part of G. By the change of variable

$$u = v + h_m$$

we can pass to the equation

$$-\Delta_g u = \lambda \left(\frac{\tilde{K}e^u}{\int_{\Sigma} \tilde{K}e^u dV_g} - \frac{1}{|\Sigma|} \right) \quad \text{in} \quad \Sigma, \tag{$*$}_\lambda$$

where

$$\tilde{K} = Ke^{-h_m}, \tag{1.6}$$

and, according to (1.3), λ is given by

$$\lambda = 4\pi(\chi(\Sigma) + \sum_{j=1}^{m} \alpha_j). \tag{1.7}$$

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