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Estimates for the kinetic transport equation in hyperbolic Sobolev spaces

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ABSTRACT

We establish smoothing estimates in the framework of hyperbolic Sobolev spaces for the velocity averaging operator ρ of the solution of the kinetic transport equation. If the velocity domain is either the unit sphere or the unit ball, then, for any exponents q and r, we find a characterisation of the exponents β_+ and β_-, except possibly for an endpoint case, for which $D_+^{\beta_+}D_-^{\beta_-}\rho$ is bounded from space–velocity $L^2_{x,v}$ to space–time $L^q_tL^r_x$. Here, D_+ and D_- are the classical and hyperbolic derivative operators, respectively. In fact, we shall provide an argument which unifies these velocity domains and the velocity averaging estimates in either case are shown to be equivalent to mixed-norm bounds on the cone multiplier operator acting on L^2. We develop our ideas further in several ways, including estimates for initial data lying in certain Besov spaces, for which a key tool in the proof is the sharp ℓ^p decoupling theorem recently established by Bourgain and Demeter. We also show that the level of permissible smoothness increases significantly if we restrict attention to initial data which are radially symmetric in the spatial variable.

1. Introduction

In this paper we consider regularity estimates for velocity integrals of the solution

$$F(x,v,t) = f(x-tv,v)$$

of the kinetic transport equation

$$(\partial_t + v\cdot\nabla)F = 0, \qquad F(x,v,0) = f(x,v),$$

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where $(x, v, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$. The regularising effect of velocity integration (or "velocity averaging") of the form

$$\rho f(x,t) = \int_{\mathbb{R}^d} f(x - tv, v)\, \mathrm{d}\mu(v) \tag{1.1}$$

for various velocity measures μ has received considerable attention in the literature, where they are often referred to as *velocity averaging lemmas* (see, for example, [7], [9], [10], [19], [20], [24], [25], [26], [34], [35], [43], [44]). Inequalities of this type are extremely rich, capturing diverse phenomena from geometric and harmonic analysis. This is perhaps most apparent through the interpretation of the dual operation

$$\rho^* g(x,v) = \int_{\mathbb{R}} g(x + tv, t)\, \mathrm{d}t \tag{1.2}$$

as a (space–time) X-ray transform, for which important problems remain wide open; see, for example, [38] or [54].

For the purposes of this introductory section, we focus our attention on the (physically-relevant) velocity average

$$\rho f(x,t) = \int_{\mathbb{S}^{d-1}} f(x - tv, v)\, \mathrm{d}\sigma(v),$$

where σ is the induced Lebesgue measure on the unit sphere $\mathbb{S}^{d-1}$. Our estimates will capture a natural regularising effect of the averaging operator ρ through the use of hyperbolic Sobolev spaces, and we begin by introducing our results in the context of initial data in $L^2(\mathbb{R}^d \times \mathbb{S}^{d-1})$. For example, given any $q, r \in [2, \infty)$, we shall obtain the optimal range of exponents β_+ and β_- (except possibly an endpoint case) for which the global space–time estimate

$$\|D_+^{\beta_+} D_-^{\beta_-} \rho f\|_{L_t^q L_x^r} \leq C \|f\|_{L^2_{x,v}} \tag{1.3}$$

holds. Here, $D_+^{\beta_+}$ denotes classical fractional differentiation of order β_+ and $D_-^{\beta_-}$ denotes the hyperbolic differentiation operator of order β_-; these are Fourier multiplier operators with multipliers $(|\xi| + |\tau|)^{\beta_+}$ and $||\xi| - |\tau||^{\beta_-}$, respectively.

As far as we are aware, Bournaveas and Perthame [16] were the first to investigate regularising properties of velocity averages over spheres using hyperbolic Sobolev spaces. They obtained (1.3) in the case $(q, r) = (2, 2)$ when $(d, \beta_+, \beta_-) = (3, \frac{1}{2}, 0)$ and $(d, \beta_+, \beta_-) = (2, \frac{1}{4}, \frac{1}{4})$. Notice that in the two-dimensional case, a *total* of $\frac{1}{2}$-derivative has been gained by the velocity average through the inclusion of hyperbolic derivatives; it was observed in [16] that such a gain is not possible by considering classical derivatives alone. These results were extended to all space dimensions $d \geq 2$ in [15] and it was shown that (1.3) holds whenever $(q, r) = (2, 2)$ and $(\beta_+, \beta_-) = (\frac{d-1}{4}, -\frac{d-3}{4})$.

We now state our first main result which gives an extension of these results to $q, r \in [2, \infty)$. In general the total number of derivatives is given by

$$\beta_+ + \beta_- = \frac{d}{r} + \frac{1}{q} - \frac{d}{2}. \tag{1.4}$$

This restriction is in fact a necessary condition for (1.3) to hold, as can be shown by a simple scaling argument. Also, it will be useful to write

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