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Global weak solutions to the compressible quantum Navier–Stokes equation and its semi-classical limit

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A B S T R A C T

This paper is dedicated to the construction of global weak solutions to the quantum Navier–Stokes equation, for any initial value with bounded energy and entropy. The construction is uniform with respect to the Planck constant. This allows to perform the semi-classical limit to the associated compressible Navier–Stokes equation. One of the difficulty of the problem is to deal with the degenerate viscosity, together with the lack of integrability on the velocity. Our method is based on the construction of weak solutions that are renormalized in the velocity variable. The existence, and stability of these solutions do not need the Mellet–Vasseur inequality.

R É S U M É

Dans ce papier, nous nous intéressons à la construction de solutions faibles pour les équations de Navier–Stokes quantiques sous l'hypothèse de conditions initiales d'énergie et d'entropie bornées. La construction étant uniforme par rapport à la constante de Planck, elle permet également de passer à la limite semi-classique vers les équations de Navier–Stokes compressibles. Une des principales difficultés du problème est de gérer à la fois la viscosité dégénérée et le manque d'intégrabilité en vitesse. Notre méthode consiste à construire des solutions faibles qui sont renormalisées en vitesse. Il est à noter que l'existence et la stabilité de ces solutions ne nécessitent pas l'inégalité de Mellet–Vasseur.

1. Introduction

Quantum models can be used to describe superfluids [12], quantum semiconductors [6], weakly interacting Bose gases [8] and quantum trajectories of Bohmian mechanics [16]. They have attracted considerable attention in the last decades due, for example, to the development of nanotechnology applications.

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In this paper, we consider the barotropic compressible quantum Navier–Stokes equations, which have been derived in [5], under some assumptions, using a Chapman–Enskog expansion in Wigner equation. In particular, we are interested in the existence of global weak solutions together with the associated semi-classical limit. The quantum Navier–Stokes equations that we are considering read as:

$$\begin{aligned}&\rho_t + \operatorname{div}(\rho u) = 0,\\ &(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma - 2\operatorname{div}(\sqrt{\nu\rho}\mathbb{S}_\nu + \sqrt{\kappa\rho}\mathbb{S}_\kappa) = \sqrt{\rho} f + \sqrt{\kappa}\operatorname{div}(\sqrt{\rho}\mathbb{M}),\end{aligned} \tag{1.1}$$

where

$$\sqrt{\nu\rho}\mathbb{S}_\nu = \nu\rho\mathbb{D}u, \qquad \operatorname{div}(\sqrt{\kappa\rho}\mathbb{S}_\kappa) = \kappa\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{1.2}$$

and with initial data

$$\rho(0,x) = \rho_0(x), \quad (\rho u)(0,x) = (\rho_0 u_0)(x) \quad \text{ in } \Omega, \tag{1.3}$$

where ρ is the density, $\gamma > 1$, $u \otimes u$ is the matrix with components $u_i u_j$, $\mathbb{D}u = \frac{1}{2}\left(\nabla u + \nabla u^T\right)$ is the symmetric part of the velocity gradient, and $\Omega = \mathbb{T}^d$ is the d-dimensional torus, here $d = 2$ or 3. The vector valued function f, and the matrix valued function $\mathbb{M}$ are source terms.

The relation (1.2) between the stress tensors and the solution $(\sqrt{\rho}, \sqrt{\rho}u)$ will be proved in the following form. For the quantic part, it will be shown that

$$2\sqrt{\kappa\rho}\mathbb{S}_\kappa = 2\kappa\left(\sqrt{\rho}\left(\nabla^2\sqrt{\rho} - 4(\nabla\rho^{1/4} \otimes \nabla\rho^{1/4})\right)\right). \tag{1.4}$$

For the viscous term, the matrix valued function $\mathbb{S}_\nu$ is the symmetric part of a matrix valued function $\mathbb{T}_\nu$, where

$$\sqrt{\nu\rho}\mathbb{T}_\nu = \nu\nabla(\rho u) - 2\nu\sqrt{\rho}u \cdot \nabla\sqrt{\rho}. \tag{1.5}$$

Whenever, ρ is regular and away from zero, the quantic part of (1.2) is equivalent to (1.4), and the matrix function $\mathbb{T}_\nu$ is formally $\sqrt{\nu\rho}\nabla u$. However, the a priori estimates do not allow to define $1/\sqrt{\rho}$ and ∇u.

The energy of the system is given by

$$E(t) = E(\sqrt{\rho}, \sqrt{\rho}u) = \int_\Omega \left(\rho\frac{|u|^2}{2} + \frac{\rho^\gamma}{\gamma - 1} + 2\kappa|\nabla\sqrt{\rho}|^2\right) dx,$$

with dissipation of entropy (in the case without source term)

$$\mathcal{D}_E(t) = \mathcal{D}_E(\mathbb{S}_\nu) = 2\int_\Omega |\mathbb{S}_\nu|^2\, dx,$$

which is formally:

$$2\nu\int_\Omega \rho|\mathbb{D}u|^2\, dx.$$

Let us recall that the energy and the dissipation of entropy satisfy the following inequality:

$$E(t) + \int_0^t \mathcal{D}_E(s) ds \leq E(0). \tag{1.6}$$

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