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Maximal functions associated with nonisotropic dilations of hypersurfaces in $\mathbb{R}^3$

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The goal of this article is to establish L^p-estimates for maximal functions associated with nonisotropic dilations of hypersurfaces in $\mathbb{R}^3$. Several results have already been obtained by Greenleaf [5], Iosevich–Sawyer [9], Ikromov–Kempe–Müller [6] and Zimmermann [28], but for some situations such as the hypersurface parameterized as the graph of a smooth function $\Phi(x_1, x_2) = x_2^d(1 + \mathcal{O}(x_2^m))$ near the origin, where $d \geq 2$, $m \geq 1$, and associated dilations $\delta_t(x) = (t^a x_1, tx_2, t^d x_3)$ for an arbitrary real number $a > 0$, the question was open until recently. In fact, such problems do arise already in lower dimensions. For instance, we consider the curve $\gamma(x) = (x, x^2(1 + \phi(x)))$ and associated dilations $\delta_t(x) = (tx_1, t^2x_2)$. If $\phi \equiv 0$, then the corresponding maximal function is the maximal function along parabolas in the plane, which is very well understood due to the work by Nagel–Riviere–Wainger [17] and others. If $\phi \neq 0$ and $\phi(x) = \mathcal{O}(x^m)$, $m \geq 1$, the problem was open until recently. We observe that in the study of the maximal function related to the mentioned curve $\gamma(x)$ and associated dilations, we will consider a family of corresponding Fourier integral operators which fail to satisfy the "cinematic curvature condition" uniformly, which means that classical local smoothing estimates established by Mockenhaupt–Seeger–Sogge could not be directly applied to our problem. In this article, we develop new ideas to establish sharp L^p-estimates for the maximal function related to the curve $\gamma(x)$ with associated dilations in the plane. Later, we generalize the result to curves of finite type d ($d \geq 2$) and associated dilations $\delta_t(x) = (tx_1, t^d x_2)$. Furthermore, we also obtain L^p-estimates for the maximal function related to the mentioned hypersurface $\Phi(x_1, x_2)$ in $\mathbb{R}^3$ with associated dilations.

R É S U M É

L'objectif de cet article est d'établir L^p-estimations pour les fonctions maximales associées aux dilatations nonisotropes des hypersurfaces dans $\mathbb{R}^3$. Plusieurs résultats ont déjà été obtenus par Greenleaf [5], Iosevich–Sawyer [9], Ikromov–Kempe–Müller [6] et Zimmermann [28], mais pour certaines situations, comme l'hypersurface paramétrée comme le graphe d'une fonction lisse $\Phi(x_1, x_2) = x_2^d(1+\mathcal{O}(x_2^m))$ près de l'origine, où $d \geq 2$, $m \geq 1$ et les dilatations associées $\delta_t(x) = (t^a x_1, tx_2, t^d x_3)$ pour un nombre réel arbitraire $a > 0$, la question était ouverte jusqu'à récemment. En fait,

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de tels problèmes se posent déjà dans des dimensions inférieures. Par exemple, nous considérons la courbe $\gamma(x) = (x, x^2(1 + \phi(x)))$ et les dilatations associées $\delta_t(x) = (tx_1, t^2x_2)$. Si $\phi \equiv 0$, alors la fonction maximale correspondante est la fonction maximale le long de paraboles dans le plan, ce qui est très bien compris grâce à l'œuvre de Nagel–Riviere–Wainger [17] et d'autres. Si $\phi \neq 0$ et $\phi(x) = \mathcal{O}(x^m)$, $m \geq 1$, le problème a été ouvert jusqu'à récemment. Nous observons que dans l'étude de la fonction maximale liée à la courbe mentionnée $\gamma(x)$ et les dilatations associées, nous considérons une famille d'opérateurs intégrales de Fourier correspondants qui ne satisfont pas à la "condition de courbure cinématographique" uniformément, ce qui veut dire que les estimations de lissage locales classiques établies par Mockenhaupt–Seeger–Sogge ne pouvaient pas être directement appliquées à notre problème. Dans cet article, nous développons de nouvelles idées pour établir des L^p-estimations pour la fonction maximale liée à la courbe $\gamma(x)$ avec des dilatations associées dans le plan. Plus tard, nous généralisons le résultat aux courbes de type fini d $(d \geq 2)$ et les dilatations associées $\delta_t(x) = (tx_1, t^dx_2)$. De plus, on obtient aussi L^p-estimations pour la fonction maximale liée à l'hypersurface $\Phi(x_1, x_2)$ dans $\mathbb{R}^3$ avec les dilatations associées.

1. Introduction

The starting point for intensive studies associated with averages over low dimensional sets is based on an earlier bound of Stein [24] from 1976 on the spherical maximal function

$$\mathcal{M}f(x) := \sup_{t>0} |M_t f(x)|,$$

where M_t are the spherical averaging operators

$$\mathcal{M}_t f(x) = \int\limits_{|y|=1} f(x - ty) d\sigma(y),$$

and $d\sigma$ is normalized surface measure on the sphere S^{n-1}. Then Stein's fundamental result shows that for $n \geq 3$, the corresponding spherical maximal operator is bounded on $L^p(\mathbb{R}^n)$ for every $p > n/(n-1)$. The analogous result in dimension two was later proved by J. Bougain [1].

Then one turned to deal with generalizations of $\mathcal{M}_t$ defined as before, i.e. the sphere is replaced by a more general smooth hypersurface $S \subset \mathbb{R}^n$. Let $\rho \in C_0^\infty(S)$ be a smooth non-negative function with compact support. Then the associated maximal operator is defined as

$$\mathcal{M}f(x) := \sup_{t>0} \left| \int\limits_S f(x - ty)\rho(y) d\mu(y) \right|, \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $d\mu$ denotes the surface measure on S. Greenleaf [5] proved that $\mathcal{M}$ is bounded on $L^p(\mathbb{R}^n)$ if $n \geq 3$ and $p > (k+1)/k$, provided S has at least $k \geq 2$ non-vanishing principal curvatures and S is starshaped with respect to the origin.

A fundamental and still largely open problem is to characterize the L^p boundedness properties of the maximal operator associated to hypersurface where the Gaussian curvature at some points is allowed to vanish. Completely understood is only the 2-dimensional case, i.e. the case of finite type curves in $\mathbb{R}^2$ studied by A. Iosevich in [7].

Many authors put a lot of effort on the development of this subject and obtained partial results in high dimension. Sogge and Stein [22] showed that if the Gaussian curvature of S does not vanish of infinite order at any point of S, then there exists a $p_0(S) < \infty$ so that the maximal function is bounded on L^p, $p > p_0(S)$.