ARTICLE IN PRESS
MATPUR:2984

J. Math. Pures Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Heat kernel estimates on connected sums of parabolic manifolds

Alexander Grigor'yan [a,1], Satoshi Ishiwata [b,*,2], Laurent Saloff-Coste [c,3]

[a] Department of Mathematics, University of Bielefeld, 33501 Bielefeld, Germany
[b] Department of Mathematical Sciences, Yamagata University, Yamagata 990-8560, Japan
[c] Department of Mathematics, Cornell University, Ithaca, NY, 14853-4201, USA

ARTICLE INFO

Article history:
Received 30 September 2016
Available online xxxx

MSC:
35K08
58J65
58J35

Keywords:
Heat kernel
Manifold with ends
Parabolic manifold
Integrated resolvent

ABSTRACT

We obtain matching two sided estimates of the heat kernel on a connected sum of parabolic manifolds, each of them satisfying the Li–Yau estimate. The key result is the on-diagonal upper bound of the heat kernel at a central point. Contrary to the non-parabolic case (which was settled in [15]), the on-diagonal behavior of the heat kernel in our case is determined by the end with the *maximal* volume growth function. As examples, we give explicit heat kernel bounds on the connected sums $\mathbb{R}^2 \# \mathbb{R}^2$ and $\mathcal{R}^1 \# \mathbb{R}^2$ where $\mathcal{R}^1 = \mathbb{R}_+ \times \mathbb{S}^1$.

RÉSUMÉ

Nous obtenons, pour la somme connexe de vraiétés riemanniennes complètes et non-compactes dont chacune satisfait l'inégalité de Li–Yau, des estimations inférieures et supérieures du noyau de la chaleur dans le cas où la variété est parabolique. Le résultat clef est l'estimation supéprieure du noyau de la chaleur sur la diagonale à un point central de la variété. Contrairement au cas non-parabolique traité dans [15], dans le cas présent, le comportement du noyau de la chaleur sur la diagonale est déterminé par le bout dont la croissance du volume est la plus forte. Parmi les exemples traités, nous donnons des estimations précises et explicites du noyau de la chaleur pour les sommes connexes $\mathbb{R}^2 \# \mathbb{R}^2$ et $\mathcal{R}^1 \# \mathbb{R}^2$ où $\mathcal{R}^1 = \mathbb{R}_+ \times \mathbb{S}^1$.

1. Introduction

Let M be a Riemannian manifold. The heat kernel $p(t,x,y)$ on M is the minimal positive fundamental solution of the heat equation $\partial_t u = \Delta u$ on M where $u = u(t,x)$, $t > 0$, $x \in M$ and Δ is the (negative

* Corresponding author.
E-mail addresses: grigor@math.uni-bielefeld.de (A. Grigor'yan), ishiwata@sci.kj.yamagata-u.ac.jp (S. Ishiwata), lsc@math.cornell.edu (L. Saloff-Coste).
[1] Partially supported by SFB 1283 of German Research Council.
[2] Partially supported by JSPS, KAKENHI 25800034, 17K05215.
[3] Partially supported by NSF grant DMS-1404435, DMS-1707589.

https://doi.org/10.1016/j.matpur.2018.03.002

definite) Laplace–Beltrami operator on M. For example, in $\mathbb{R}^n$ the heat kernel is given by the classical Gauss–Weierstrass formula

$$p(t,x,y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right).$$

The heat kernel is sensitive to the geometry of the underlying manifold M, which results in numerous applications of this notion in differential geometry. On the other hand, the heat kernel has a probabilistic meaning: $p(t,x,y)$ is the transition density of Brownian motion $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ on M. Namely, for any Borel set $A \subset M$, we have

$$\mathbb{P}_x(X_t \in A) = \int_A p(t,x,y)dy,$$

where $\mathbb{P}_x(X_t \in A)$ is the probability that Brownian particle starting at the point x will be found in the set A in time t.

From now on let us assume that the manifold M is non-compact and geodesically complete. Dependence of the long time behavior of the heat kernel on the large scale geometry of M is an interesting and important problem that has been intensively studied during the past few decades by many authors (see, for example, [4], [10], [21] and references therein). In the case when the Ricci curvature of M is non-negative, P. Li and S.-T. Yau proved in their pioneering work [19] the following estimate, for all $x, y \in M$ and $t > 0$:

$$p(t,x,y) \asymp \frac{C}{V(x,\sqrt{t})} \exp\left(-b\frac{d^2(x,y)}{t}\right), \tag{1.1}$$

where the sign $\asymp$ means that both $\leq$ and $\geq$ hold but with different values of positive constants C and b, $V(x,r)$ is the Riemannian volume of the geodesic ball of radius r centered at $x \in M$, and $d(x,y)$ is the geodesic distance between the points x, y.

The estimate (1.1) is satisfied also for the heat kernel of uniformly elliptic operators in divergence form in $\mathbb{R}^n$ as was proved by Aronson [1]. It was proved by Fabes and Stroock [6], that the estimate (1.1) is equivalent to the uniform parabolic Harnack inequality (see also [21]). Grigor'yan [7] and Saloff-Coste [20], [21] proved that (1.1) is equivalent to the conjunction of the Poincaré inequality and the volume doubling property.

One of the simplest example of a manifold where (1.1) fails is the hyperbolic space $\mathbb{H}^n$. A more interesting counterexample was constructed by Kuz'menko and Molchanov [18]: they showed that the connected sum $\mathbb{R}^n \# \mathbb{R}^n$ of two copies of $\mathbb{R}^n$, $n \geq 3$, admits a non-trivial bounded harmonic function, which implies that the Harnack inequality and, hence, (1.1) cannot be true. Benjamini, Chavel and Feldman [2] explained this phenomenon by a bottleneck-effect: if x and y belong to the different ends of the manifold $\mathbb{R}^n \# \mathbb{R}^n$ and $|x| \approx |y| \approx \sqrt{t} \to \infty$ then $p(t,x,y) \ll t^{-n/2}$ where $t^{-n/2}$ is predicted by the right hand side of (1.1). This phenomenon is especially transparent from probabilistic viewpoint: Brownian particle can go from x to y only through the central part, which reduces drastically the transition density (see Fig. 1). A similar phenomenon was observed by B. Davies [5] on a model case of one-dimensional line complex.

Based on these early works, the first and the third authors of the present paper started a project on heat kernel bounds on connected sums of manifolds, provided each of them satisfies the Li–Yau estimate (1.1). The results of this study are published in a series [11], [12], [13], [15], and [16]. In particular, they obtained in [15] matching upper and lower estimates of heat kernels on connected sums of manifolds when at least one of them is *non-parabolic*. Recall that a manifold M called *parabolic* if Brownian motion on M is recurrent, and *non-parabolic* otherwise. There are several equivalent definitions of parabolicity in different terms (see, for example, [9]).

Please cite this article in press as: A. Grigor'yan et al., Heat kernel estimates on connected sums of parabolic manifolds, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.03.002