ARTICLE IN PRESS

MATPUR:2977

J. Math. Pures Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal de Mathématiques Pures et Appliquées

www.elsevier.com/locate/matpur

Embedded eigenvalues of the Neumann problem in a strip with a box-shaped perturbation

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ARTICLE INFO

Article history:
Received 5 February 2015
Available online xxxx

MSC:
35P05
47A75
49R50
78A50

Keywords:
Acoustic waveguide
Neumann problem
Embedded eigenvalues
Continuous spectrum
Box-shaped perturbation
Asymptotics

ABSTRACT

We consider the spectral Neumann problem for the Laplace operator in an acoustic waveguide Π_l^ε formed by the union of an infinite strip and a narrow box-shaped perturbation of size $2l \times \varepsilon$, where $\varepsilon > 0$ is a small parameter. We prove the existence of the length parameter $l_k^\varepsilon = \pi k + O(\varepsilon)$ with any $k = 1, 2, 3, \ldots$ such that the waveguide $\Pi_{l_k^\varepsilon}^\varepsilon$ supports a trapped mode with an eigenvalue $\lambda_k^\varepsilon = \pi^2 - 4\pi^4 l^2 \varepsilon^2 + O(\varepsilon^3)$ embedded into the continuous spectrum. This eigenvalue is unique in the segment $[0, \pi^2]$, and it is absent in the case $l \neq l_k^\varepsilon$. The detection of this embedded eigenvalue is based on a criterion for trapped modes involving an artificial object, the augmented scattering matrix. The main difficulty is caused by the rather specific shape of the perturbed wall $\partial\Pi_l^\varepsilon$, namely a narrow rectangular bulge with corner points, and we discuss available generalizations for other piecewise smooth boundaries.

RÉSUMÉ

Nous considérons le spectre du laplacien de Neumann dans un guide d'onde acoustique Π_l^ε formé d'une bande infinie perturbée par une cavité étroite de taille $2l \times \varepsilon$, où $\varepsilon > 0$ est un petit paramètre. Nous prouvons, pour chaque $k = 1, 2, 3$, l'existence d'une longueur $l_k^\varepsilon = \pi k + O(\varepsilon)$ telle qu'il existe dans le guide d'onde $\Pi_{l_k^\varepsilon}^\varepsilon$ un mode piégé associé à une valeur propre $\lambda_k^\varepsilon = \pi^2 - 4\ pi^4 l^2 \varepsilon^2 + O(\varepsilon^3)$ plongée dans le spectre continu. Il s'agit de la seule valeur propre dans $[0, \pi^2]$, elle est de plus absente lorsque $l \neq l_k^\varepsilon$. La détection de cette valeur propre plongée est basée sur un critère pour les modes piégés impliquant un objet artificiel : la matrice de scattering augmentée. La principale difficulté vient de la forme spécifique de la perturbation du mur $\partial\Pi_l^\varepsilon$, c'est-à-dire un étroit renflement rectangulaire comportant des angles. Nous discutons également de possibles généralisations aux cas d'autres bords constants par morceaux.

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https://doi.org/10.1016/j.matpur.2018.01.002
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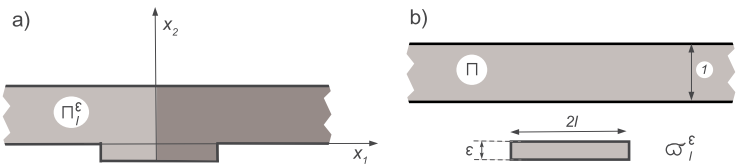

Fig. 1. The waveguide with a box-shaped perturbation (a) and its fragments (b).

1. Introduction

1.1. Formulation of the problems

In the union Π_l^ε, Fig. 1, a and b, of the straight unit strip

$$\Pi = \left\{x = (x_1, x_2) \in \mathbb{R}^2,\ x_1 \in \mathbb{R},\ x_2 \in (0,1)\right\} \tag{1.1}$$

and a rectangle of length $2l > 0$ and small width $\varepsilon > 0$,

$$\varpi_l^\varepsilon = \{x : |x_1| < l,\quad x_2 \in (-\varepsilon, 0]\}, \tag{1.2}$$

we consider the spectral Neumann problem

$$-\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x),\quad x \in \Pi_l^\varepsilon = \Pi \cup \varpi_l^\varepsilon, \tag{1.3}$$

$$\partial_\nu u^\varepsilon(x) = 0,\ x \in \partial\Pi_l^\varepsilon, \tag{1.4}$$

where $\Delta = \nabla \cdot \nabla$ is the Laplace operator, $\nabla = \text{grad}$, λ^ε is the spectral parameter and $\partial_\nu = \nu \cdot \nabla$ is the directional derivative, ν stands for the unit outward normal defined everywhere at the boundary $\partial\Pi_l^\varepsilon$, except for the corner points, i.e., the vertices of the rectangle (1.2). Since a solution of the problem (1.3), (1.4) may get singularities at these points, the problem ought to be reformulated as the integral identity [43]

$$(\nabla u^\varepsilon, \nabla v^\varepsilon)_{\Pi_l^\varepsilon} = \lambda^\varepsilon (u^\varepsilon, v^\varepsilon)_{\Pi_l^\varepsilon}\quad \forall v^\varepsilon \in H^1(\Pi_l^\varepsilon), \tag{1.5}$$

where $(\ ,\ )_{\Pi_l^\varepsilon}$ is the natural scalar product in the Lebesgue space $L^2(\Pi_l^\varepsilon)$ and $H^1(\Pi_l^\varepsilon)$ stands for Sobolev space. The symmetric bilinear form on the left-hand side of (1.5) is closed and positive in $H^1(\Pi_l^\varepsilon)$ so that the problem (1.3), (1.4) is associated [6, Ch. 10] with a positive self-adjoint operator A_l^ε in $L^2(\Pi_l^\varepsilon)$ whose spectrum $\wp = \wp_{co}$ is continuous and covers the closed positive semi-axis $\overline{\mathbb{R}}_+ = [0, +\infty)$. The domain $\mathcal{D}(A_l^\varepsilon)$ of A_l^ε, of course, is contained in $H^1(\Pi_l^\varepsilon)$ but it is bigger than $H^2(\Pi_l^\varepsilon)$ due to singularities of solutions at the corner points, see, e.g., [59, Ch. 2]. The discrete spectrum of the operator A_l^ε clearly is empty but its point spectrum $\wp_{po}$, consisting of embedded eigenvalues inside the continuous spectrum, can be non-empty. The main goal of our paper is to single out a particular value of the length parameter l such that the operator A_l^ε gets an eigenvalue $\lambda_l^\varepsilon \in \wp_{po}$ embedded into the continuous spectrum $\wp_{co}$. The corresponding eigenfunction $u_l^\varepsilon \in H^1(\Pi_l^\varepsilon)$ decays exponentially at infinity and is called a trapped mode, cf. [44] and [63]. We use "trapped mode" as a synonym of "eigenfunction" throughout the paper.

Our main result, formulated below in Theorem 3, roughly speaking, demonstrates that an eigenvalue λ_l^ε exists in the interval $\left(0, \pi^2\right) \subset \wp_{co}$ for $l^\varepsilon \approx \pi k$ with $k \in \mathbb{N} = \{1, 2, 3, ...\}$ only. To obtain this result, we in particular construct asymptotics of λ^ε and l^ε as $\varepsilon \to +0$.

The problem (1.3), (1.4) is a model of an acoustic waveguide with hard walls, cf. [49], but is also related in a natural way to the linear theory of surface water-waves, cf. [42]. Indeed, the velocity potential $\Phi^\varepsilon(x, z)$ satisfies the Laplace equation in the channel $\Xi_{l,d}^\varepsilon = \Pi_l^\varepsilon \times (-d, 0) \subset \mathbb{R}^3 \ni (x, z)$ of depth $d > 0$ with the

Please cite this article in press as: G. Cardone et al., Embedded eigenvalues of the Neumann problem in a strip with a box-shaped perturbation, J. Math. Pures Appl. (2018), https://doi.org/10.1016/j.matpur.2018.01.002