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Singular structure formation in a degenerate haptotaxis model involving myopic diffusion

Michael Winkler

Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany

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ABSTRACT

We consider the system

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \big(d(x)uw_x\big)_x, \\ w_t = -ug(w), \end{cases} \tag{0.1}$$

which arises as a simple model for haptotactic migration in heterogeneous environments, such as typically occurring in the invasive dynamics of glioma. A particular focus is on situations when the diffusion herein is degenerate in the sense that the zero set of d is not empty.
It is shown that if such possibly present degeneracies are sufficiently mild in the sense that

$$\int_\Omega \frac{1}{d} < \infty, \tag{0.2}$$

then under appropriate assumptions on the initial data a corresponding initial-boundary value problem for (0.1), posed under no-flux boundary conditions in a bounded open interval $\Omega \subset \mathbb{R}$, possesses at least one globally defined generalized solution.
Moreover, despite such degeneracies the myopic diffusion mechanism in (0.1) is seen to asymptotically determine the solution behavior in the sense that for some constant $\mu_\infty > 0$, the obtained solution satisfies

$$\begin{aligned} &u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d} \quad \text{in } L^1(\Omega) \qquad \text{and} \\ &w(\cdot,t) \to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } t \to \infty, \end{aligned} \tag{0.3}$$

and that hence in the degenerate case the solution component u stabilizes toward a state involving infinite densities, which is in good accordance with experimentally observed phenomena of cell aggregation.
Finally, under slightly stronger hypotheses inter alia requiring that $\frac{1}{d}$ belong to $L\log L(\Omega)$, a substantial effect of diffusion is shown to appear already immediately by proving that for a.e. $t > 0$, the quantity $\ln(du(\cdot,t))$ is bounded in Ω. In degenerate

E-mail address: michael.winkler@math.uni-paderborn.de.

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situations, this particularly implies that the blow-up phenomena expressed in (0.3) in fact occur instantaneously.

R É S U M É

Nous consiérons le système

$$\begin{cases} u_t = \big(d(x)u\big)_{xx} - \big(d(x)uw_x\big)_x, \\ w_t = -ug(w), \end{cases} \tag{0.1}$$

qui est issu d'un modèle simplifié pour la migration hapto-tactique dans des milieux hétérogènes, comme par exemple dans la dynamique invasive des gliomes. Un intérêt particulier est porté aux situations pour lesquelles la diffusion est dégénérée, c'est-à-dire que le noyau de d est non vide.
À (0.1) on peut faire correspondre un problème aux limites, posé sur intervalle ouvert et borné $\Omega \subset \mathbb{R}$, associé à une condition de flux nul au bord. Il est montré que si les éventuelles dégénérescences de d sont suffisament modérées au sens que

$$\int_\Omega \frac{1}{d} < \infty, \tag{0.2}$$

alors il existe au moins une solution généralisée au problème, définie globalement. Ce, sous des hypothèses appropriées sur la donnée initiale.
De plus, malgré ces dégénérescences, le mécanisme de diffusion myopique de (0.1) determine le comportement asymptotique de la solution. Plus précisément, pour une certaine constante $\mu_\infty > 0$, la solution obtenue satisfait

$$\begin{aligned} &u(\cdot,t) \rightharpoonup \frac{\mu_\infty}{d} \quad \text{dans } L^1(\Omega) \qquad \text{et} \\ &w(\cdot,t) \to 0 \quad \text{dans } L^\infty(\Omega) \qquad \text{pour } t \to \infty. \end{aligned} \tag{0.3}$$

Donc dans le cas dégénéré, la composante u de la solution converge vers un état présentant des densités infinies. Ceci est en adéquation avec les phénomènes d'aggrégation cellulaire observés expérimentalement.
Finalement, sous des hypothèses un peu plus fortes, dont la contrainte $d \in L\log L(\Omega)$, il est montré que la diffusion a un effet presque immédiat. C'est-à-dire que pour presque tout $t > 0$, la quantité $\ln(du(\cdot,t))$ est bornée dans Ω. Dans le cas dégénéré, ceci implique en particulier que les phénomènes d'explosion décrits par (0.3) ont lieu instantanément.

1. Introduction

In the theoretical description of collective cell behavior at macroscopic scales, taxis mechanisms have been playing an increasingly substantial role [18]. In the past two decades, an accordingly growing literature on mathematical analysis of such processes has brought about quite a thorough knowledge of various classes of corresponding PDE models, containing cross-diffusive parabolic equations as their most characteristic ingredient, especially in situations when the attractive signal is a chemical and hence diffusible (see [2] for a recent survey). Unlike such chemotaxis systems, considerably less understood seem so-called haptotaxis systems which substantially differ from the former in that they address cases of non-diffusible cues, as naturally involved when tumors invade healthy tissue. Moreover, virtually all analytical studies on taxis systems assume that random movement of cells is of Fickian diffusion type, either linear or nonlinear, with few exceptions considering fractional diffusion chemotaxis models [6,7]. Recent modeling approaches, however, indicate that in situations of significantly heterogeneous environments, adequate macroscopic limits