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# On the spectral properties of Dirac operators with electrostatic $\delta$-shell interactions

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ABSTRACT

In this paper the spectral properties of Dirac operators $A_\eta$ with electrostatic $\delta$-shell interactions of constant strength $\eta$ supported on compact smooth surfaces in $\mathbb{R}^3$ are studied. Making use of boundary triple techniques a Krein type resolvent formula and a Birman–Schwinger principle are obtained. With the help of these tools some spectral, scattering, and asymptotic properties of $A_\eta$ are investigated. In particular, it turns out that the discrete spectrum of $A_\eta$ inside the gap of the essential spectrum is finite, the difference of the third powers of the resolvents of $A_\eta$ and the free Dirac operator $A_0$ is trace class, and in the nonrelativistic limit $A_\eta$ converges in the norm resolvent sense to a Schrödinger operator with an electric $\delta$-potential of strength $\eta$.


RÉSUMÉ

Dans cet article, on étudie les propriétés spectrales des opérateurs de Dirac $A_\eta$ avec une interaction $\delta$ électrostatique de force constante $\eta$ supportée sur des surfaces compactes régulières dans $\mathbb{R}^3$. En utilisant des techniques de triplets au bord, une formule de résolvante à la Krein et un principe de Birman–Schwinger sont obtenus. Grâce à ces outils, certaines propriétés spectrales, de diffusion et asymptotiques de $A_\eta$ sont étudiées. En particulier, il s'avère que le spectre discret de $A_\eta$ dans le trou du spectre essentiel est fini, les différences entre les puissances troisièmes des résolvantes de $A_\eta$ et $A_0$ sont de classe trace, et dans la limite non relativiste $A_\eta$ converge en norme de la résolvante vers un opérateur de Schrödinger avec une interaction $\delta$ électrique de force $\eta$.


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## 1. Introduction

Singular $\delta$-interactions are often used as idealized replacements for strongly localized electric potentials; the spectral data, the scattering properties, and the location of resonances for the original operator can be deduced then approximately. While Schrödinger operators with $\delta$-interactions supported on manifolds of small co-dimensions were investigated extensively, cf. the monographs [1,11,22] and the review article [21], much less attention was paid to Dirac operators with $\delta$-interactions.

Let us choose units such that $\hbar = 1$ and denote the speed of light by $c$. It is well-known that the free Dirac operator

$$A_0 := -ic\sum_{j=1}^{3}\alpha_j\partial_j + mc^2\beta = -ic\alpha\cdot\nabla + mc^2\beta, \qquad \operatorname{dom} A_0 = H^1(\mathbb{R}^3;\mathbb{C}^4),$$

where $m > 0$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\beta$ denote the Dirac matrices described in (1.1) below, is self-adjoint in $L^2(\mathbb{R}^3;\mathbb{C}^4)$ and that

$$\sigma(A_0) = (-\infty, -mc^2] \cup [mc^2, \infty).$$

The free Dirac operator describes the motion of a spin-$\frac{1}{2}$ particle with mass $m$ in vacuum taking relativistic aspects into account; cf. [32]. In the following let $\Sigma$ be the boundary of a bounded $C^\infty$-smooth domain $\Omega \subset \mathbb{R}^3$. Then the Dirac operator with an electrostatic $\delta$-shell interaction supported on $\Sigma$ with a constant interaction strength $\eta \in \mathbb{R}$ is formally given by

$$A_\eta = -ic\alpha\cdot\nabla + mc^2\beta + \eta\delta_\Sigma,$$

where $\delta_\Sigma$ stands for the $\delta$-distribution supported on the surface $\Sigma$ acting as

$$\delta_\Sigma f = \frac{1}{2}\big(f_+|_\Sigma + f_-|_\Sigma\big); \quad f_+ = f|_\Omega,\ f_- = f|_{\mathbb{R}^3\setminus\overline{\Omega}}.$$

Note that $A_\eta$ is defined on functions that are weakly differentiable away from $\Sigma$, the $\delta$-interaction is then modeled, as usual, by a jump condition for these functions on $\Sigma$. It is the main objective of this paper to analyze the properties of Dirac operators with electrostatic $\delta$-shell interactions by applying the abstract technique of quasi boundary triples and their Weyl functions from extension theory of symmetric operators. Our investigations and some of our results are inspired by the very recent contributions [2–4] in this area.

The mathematical study of Dirac operators with $\delta$-interactions started in the 1980s. One-dimensional Dirac operators with singular point interactions were studied in [25]; cf. also [1, Appendix J], [15] and the references therein, and the first mathematically rigorous treatise on a Dirac operator in $\mathbb{R}^3$ with a $\delta$-shell interaction supported on a sphere was [19]. Using a decomposition into spherical harmonics and the results on the one-dimensional Dirac operator with singular interactions the self-adjointness of $A_\eta$ and a number of spectral properties were shown in [19]. The interest in the topic arose again with the discovery of a family of artificial materials where the Dirac equation can be approximately deduced from Schrödinger's equation [33]. From a mathematical point of view the investigation of Dirac operators with $\delta$-interactions supported on more general surfaces in $\mathbb{R}^3$ was initiated recently in [2–4].

Our motivation is to show how the concept of quasi boundary triples and their Weyl functions can be used to introduce and study Dirac operators with electrostatic $\delta$-shell interactions. Quasi boundary triples are a slight generalization of the concept of (ordinary) boundary triples, which is a powerful tool in the analysis of self-adjoint extensions of symmetric operators [13,14,17,27,29]. Quasi boundary triples were originally introduced in [6] for the study of elliptic partial differential operators, they were applied in the

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