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Regularizing effect of L^q interplay between coefficients in some elliptic equations ☆

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A B S T R A C T

In [1], even if f only belongs to $L^1(\Omega)$, we proved that the assumption $|f(x)| \leq Q\,a(x) \in L^1(\Omega)$, $Q \in \mathbb{R}^+$, implies the existence of a bounded solution of boundary value problems, whose simplest example is the linear problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)\,u = f(x), \text{ in } \Omega; \\ \qquad\qquad\qquad\qquad\qquad u = 0, \text{ on } \partial\Omega; \end{cases}$$

where Ω is a bounded open set of $\mathbb{R}^N$ and M is a bounded elliptic matrix.
In this paper, we continue the study of the regularizing effect on the solutions, in some nonlinear Dirichlet problems, when we have L^q interaction between the coefficient $a(x)$ and the datum $f(x)$ (i.e. Q is now a function in $L^q(\Omega)$).

R É S U M É

Dans l'article [1], même avec $f \in L^1(\Omega)$, on demontre que $|f(x)| \leq Q\,a(x) \in L^1(\Omega)$, $Q \in \mathbb{R}^+$, implique l'existence d'une solution bornée de problèmes elliptiques, dont l'exemple plus simple est le problème lineaire

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)\,u = f(x), \text{ dans } \Omega; \\ \qquad\qquad\qquad\qquad\qquad u = 0, \text{ sur } \partial\Omega \end{cases}$$

avec Ω un ouvert borné de $\mathbb{R}^N$ et M une matrice bornée et elliptique.
Dans cet article, on poursuit l'étude de l'effet régularisant sur les solutions de problèmes elliptiques non linéaires si on a une interaction L^q entre le coefficient $a(x)$ et $f(x)$ (c'est-à-dire si Q est une fonction de $L^q(\Omega)$).

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1. Introduction

This paper concerns the interplay between coefficients in nonlinear Dirichlet problems. The simplest case is the linear boundary value problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)\,u = f(x), \text{ in } \Omega; \\ \qquad\qquad\qquad\qquad u = 0, \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^N$, $N > 2$, and for some positive constants α, β, the bounded elliptic matrix $M(x) = (m_{i,j}(x))_{1\le i,j\le N}$ satisfies ($x \in \Omega$, $\xi \in \mathbb{R}^N$),

$$\alpha|\xi|^2 \le M(x)\,\xi\,\xi, \qquad |\,M(x)| \le \beta\,, \tag{1.2}$$

where $|M(x)|$ denotes $\sup_{1\le i,j\le N} |m_{i,j}(x)|$. We note that it is possible to assume $N \ge 2$ with some changes, since then 2^* is not well defined. The main result in [1] states that if we assume $a(x), f(x) \in L^1(\Omega)$ and that there exists $Q \in \mathbb{R}^+$ such that

$$|f(x)| \le Q\,a(x), \tag{1.3}$$

then there exists a weak solution $u \in W_0^{1,2}(\Omega)$ of (1.1) such that $\|u\|_{L^\infty(\Omega)} \le Q$.

Thus it is proved that the interplay between the coefficient of the lower-order term and the right-hand side yields some regularizing effects on the solution, so that the lower-order term is not so "lower" and the principal part is not completely principal in this study. The previous sentence also applies (with the same proof) for problems whose principal part is nonlinear; we state again the existence results of [1] in a nonlinear framework in the last section.

The aim of this paper is to continue the study of more general interplays between both coefficients $a(x), f(x) \in L^1(\Omega)$ with $a(x) \ge 0$. Specifically, in Section 2 we prove the existence of solutions in $W_0^{1,2}(\Omega)$ under the assumption that there exist $Q \in \mathbb{R}^+$ and $0 \le R(x) \in L^{\frac{2N}{N+2}}(\Omega)$ satisfying

$$|f(x)| \le Q\,a(x) + R(x). \tag{1.4}$$

When M is depending also on u and there is a nonlinear lower-order term with quadratic growth on $|\nabla u|$ the existence of solution in $W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ is obtained in Section 3 provided that the function $R(x) \in L^m(\Omega)$, $m > N/2$. Observe that in particular, (1.4) holds if there exist $Q \in \mathbb{R}^+$ and $\theta < 1$ such that

$$|f(x)| \le Q\,a(x)^\theta. \tag{1.5}$$

On the other hand, the case that

$$|f(x)| \le a(x)^\lambda, \quad \lambda > 1, \quad a(x) \in L^p(\Omega), \tag{1.6}$$

is studied in Section 4. It is proved there the existence of a solution of (1.1)

- in $W_0^{1,2}(\Omega)$ when $p \ge 2\lambda - 1$;
- in $W_0^{1,r}(\Omega)$, $r = (\frac{1}{N} + \frac{2(\lambda-1)}{2^*(p-1)})^{-1}$, if $2\lambda - 1 > p > \lambda$.

Again this is a corollary of a more general result whose motivation is to handle the case of non-constant functions $Q(x)$ in (1.3). More specifically, existence of solutions is discussed when it is satisfied the condition