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Inversion formulas and range characterizations for the attenuated geodesic ray transform

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A B S T R A C T

We present two range characterizations for the attenuated geodesic X-ray transform defined on pairs of functions and 1-forms on simple surfaces. Such characterizations are based on first isolating the range over sums of functions and 1-forms, then separating each sub-range in two ways, first by implicit conditions, second by deriving new inversion formulas for sums of functions and 1-forms.

R É S U M É

Nous présentons deux caractérisations de l'image de la transformée Rayons X géodésique atténuée, définie pour des sommes de fonctions et 1-formes, sur des surfaces à bord dites « simples ». Ces caractérisations sont obtenues en isolant d'abord l'image de l'opérateur aux sommes de fonctions et 1-formes, puis en caractérisant les restrictions aux fonctions ou 1-formes de deux manières, d'une part via des conditions implicites, d'autre part en dérivant des nouvelles formules de reconstruction pour les fonctions et 1-formes à partir de leurs transformées Rayons X.

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1. Introduction

Let (M, g) be a smooth compact oriented Riemannian surface with boundary ∂M, with unit tangent bundle $SM := \{(x, v) \in TM : |v|_{g(x)} = 1\}$ and inward/outward boundaries

$$\partial_\pm SM = \{(x, v) \in SM : x \in \partial M,\ \pm\langle v, \nu_x\rangle_{g(x)} \geq 0\},$$

where ν_x is the unit inward normal at $x \in \partial M$. Denote $\varphi_t : SM \to SM$ the geodesic flow, written as $\varphi_t(x, v) = (\gamma_{x,v}(t), \dot\gamma_{x,v}(t))$ and defined for $-\tau(x, -v) \leq t \leq \tau(x, v)$, where $\tau(x, v)$ is the first exit time of the geodesic starting at (x, v). Throughout the paper, we assume that (M, g) is *simple*, meaning that the boundary is strictly convex and that any two points on the boundary are joined by a unique minimizing geodesic. In particular, this implies that (M, g) is simply connected (see, e.g. [20, Proposition 2.4]) and that $\tau(x, v)$ is bounded on SM (i.e., (M, g) is non-trapping). For $a \in C^\infty(M, \mathbb{C})$, the object of study is the *attenuated geodesic ray transform* $I_a : C^\infty(SM, \mathbb{C}) \to C^\infty(\partial_+SM, \mathbb{C})$ defined for $f \in C^\infty(SM, \mathbb{C})$ as

$$I_a f(x, v) = \int_0^{\tau(x,v)} f(\varphi_t(x, v)) \exp\left(\int_0^t a(\gamma_{x,v}(s))\, ds\right) dt, \quad (x, v) \in \partial_+SM, \tag{1}$$

where the definition follows the convention in [29].[3] The present article aims at providing range characterizations for this transform over pairs of functions and 1-forms, that is, when the integrand f in (1) takes the form $f(x, v) = f_0(x) + \boldsymbol{\alpha}_x(v)$ for f_0 a function and $\boldsymbol{\alpha}$ a 1-form. As the transform above models some medical imaging modalities such as Computerized Tomography and Ultrasound Doppler Tomography in media with variable refractive index, range characterizations are useful to project noisy data onto the range of a given measurement operator before inverting for the unknown (f_0 or $\boldsymbol{\alpha}$ here). In media with constant refractive index, modeled by the Euclidean metric in the parallel geometry, the problem has been extensively studied [18,16,2,6,34]. In this setting, a range characterization has been obtained in [17], though obtaining such a characterization in the form of Helgason–Ludwig type consistency conditions, a form enjoyed by the unattenuated transform which is most amenable to practical data denoising, remains an open problem. Recently, range characterizations for the attenuated transform on convex Euclidean domains were provided in terms of Hilbert transforms with respect to A-analytic function theory *à la* Bukhgeim, treating the case of functions [28], vector fields [27] and two-tensors [26], though such results are limited to Euclidean settings, as the theory of A-analytic functions has not yet been developed on non-Euclidean domains.

In the case of manifolds with no symmetries, parallel geometry does not exist and one must work with fan-beam coordinates. The scalar case has been studied in [29,14] in the geodesic case, and in [11] in the Euclidean, fan-beam case, mainly focused on injectivity, stability and inversion procedures.

On to range characterizations, the first one in terms of boundary operators was provided by Pestov and Uhlmann in [23], later generalized to the case of transport with unitary connection, with further applications to the range characterization of the unattenuated transform over higher-order tensors [19]. Recently in [15], the range characterization in [23] was proved by the second author to be a generalization of the classical moment conditions in the Euclidean setting.

In the approach coming from [23], there exists a boundary operator P which only depends on the scattering relation and the fiberwise Hilbert transform, and which characterizes the unattenuated transform over functions and 1-forms. Further splitting of P into the sum $P_+ + P_-$ allows to separate ranges over functions and 1-forms. A major challenge in the attenuated case is that, despite the fact that a similar

[3] In the applied literature, the attenuation term inside the integrand may take the form $\exp\left(-\int_t^{\tau(x,v)} a(\gamma_{x,v}(s))\, ds\right)$ instead. Both transforms obtained differ by a multiplicative factor $\exp(I_0 a)$ and thus contain the same information provided the attenuation a is known.

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