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Homogenization of pathwise Hamilton–Jacobi equations

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A B S T R A C T

We present qualitative and quantitative homogenization results for pathwise Hamilton–Jacobi equations with "rough" multiplicative driving signals. When there is only one such signal and the Hamiltonian is convex, we show that the equation, as well as equations with smooth approximating paths, homogenize. In the multi-signal setting, we demonstrate that blow-up or homogenization may take place. The paper also includes a new well-posedness result, which gives explicit estimates for the continuity of the solution map and the equicontinuity of solutions in the spatial variable.

R É S U M É

Nous présentons des résultats qualitatifs et quantitatifs d'homogénéisation pour des équations de Hamilton–Jacobi avec des termes de forçage multiplicatifs irréguliers. Dans le cas d'un seul forçage et d'un Hamiltonien convexe, nous montrons que l'équation elle-même, et les équations qui l'approchent pour des forçages réguliers, s'homogénéisent. Dans le cas de plusieurs forçages, nous prouvons que l'explosion ou l'homogénéisation sont possibles. L'article contient aussi un nouveau résultat sur le caractère bien posé de l'équation, qui donne des estimées explicites sur la continuité de l'application qui aux termes de forçages et à la condition initiale associe la solution, ainsi que sur l'équicontinuité des solutions par rapport à la variable d'espace.

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1. Introduction

We study the asymptotic behavior, as $\epsilon \to 0$, of pathwise Hamilton–Jacobi equations driven by a "rough" continuous signal $W : [0, \infty) \to \mathbb{R}^M$,

$$du^\epsilon + \sum_{i=1}^{M} H^i(Du^\epsilon, x/\epsilon) \cdot dW^i = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty), \qquad u^\epsilon(\cdot, 0) = u_0 \quad \text{on } \mathbb{R}^d. \tag{1.1}$$

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The initial condition u_0 belongs to $BUC(\mathbb{R}^d)$, the space of bounded, uniformly continuous functions on $\mathbb{R}^d$. We expect homogenization to occur if the dependence of H^i in the spatial variable $y = x/\epsilon$ has some sort of self-averaging property. In this paper, this will be periodicity or stationary ergodicity.

The interpretation of (1.1) is determined by the regularity of W. For example, if W is differentiable, then dW and du^ϵ represent respectively the time derivatives of u^ϵ and W, which we denote by $u^\epsilon_t = u^\epsilon_t(x,t)$ and $\dot{W} = \dot{W}_t = \dot{W}(t)$. In this case, u^ϵ is defined in the usual Crandall–Lions viscosity sense, the theory for which is outlined in the User's Guide by Crandall, Ishii, and Lions [8]. If W has bounded variation, then (1.1) falls within the scope of equations with L^1-time dependence considered by Lions and Perthame [17] and Ishii [13]. In either setting, the symbol $\cdot$ stands for multiplication.

Here, we allow W to be any continuous signal. The typical examples are sample paths of continuous stochastic processes, such as Brownian motion, in which case W is nowhere differentiable and has unbounded variation on every interval. For such W, the symbol $\cdot$ should be thought of as the Stratonovich differential.

The theory for (1.1) in this generality was developed by Lions and Souganidis in [19], [20], [21], and [22], and is discussed in more detail in the forthcoming book [23]. Proving well-posedness is more challenging than in the classical viscosity setting, especially for spatially dependent H^i. In general, strong regularity is required for the Hamiltonians, and (1.1) is only well-posed for W in certain Hölder spaces.

In the single path case $M = 1$, if the Hamiltonian is uniformly convex, one can weaken the regularity assumptions and prove well-posedness for any continuous W. This is discussed by Lions and Souganidis in a forthcoming work [18], and a specific example is considered by Friz, Gassiat, Lions, and Souganidis in [11].

We briefly outline the results proved in this paper, giving the precise statements later. The various assumptions, including the homogenization rate assumption (3.8) that we reference below, are listed in Section 3.

We first study the single path setting,

$$du^\epsilon + H(Du^\epsilon, x/\epsilon) \cdot dW = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \qquad u^\epsilon(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^d. \tag{1.2}$$

Following [18], we prove the following new well-posedness result.

Theorem 1.1. *Assume that H is smooth, uniformly convex, and, for some $q' > 1$, positively homogenous of degree q'. Then, for all $\epsilon > 0$, $u_0 \in BUC(\mathbb{R}^d)$, and $W \in C([0,\infty),\mathbb{R})$, (1.2) admits a unique pathwise viscosity solution in the sense of Lions and Souganidis. Moreover, the solution operator for (1.2) is uniformly continuous in W, and the modulus of continuity for $u^\epsilon(\cdot,t)$ depends only on the growth of H.*

Using Theorem 1.1, we prove that, as $\epsilon \to 0$, u^ϵ converges to the unique solution of a homogenized equation of the form

$$du + \overline{H}(Du) \cdot dW = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \qquad u(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^d. \tag{1.3}$$

Theorem 1.2. *In addition to the hypotheses of Theorem 1.1, assume (3.8). Then there exists $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ such that, for all $u_0 \in BUC(\mathbb{R}^d)$ and $W \in C([0,\infty),\mathbb{R})$, the solution u^ϵ of (1.2) converges locally uniformly, as $\epsilon \to 0$, to the solution u of (1.3). Moreover, the convergence is uniform over all u_0 with uniformly bounded Lipschitz constant.*

For more general Hamiltonians, we replace W with a smooth signal W^ϵ that converges locally uniformly, as $\epsilon \to 0$, to W, and study the initial value problem

$$u^\epsilon_t + H(Du^\epsilon, x/\epsilon)\dot{W}^\epsilon_t = 0 \quad \text{in } \mathbb{R}^d \times (0,\infty), \qquad u^\epsilon(\cdot,0) = u_0 \quad \text{on } \mathbb{R}^d. \tag{1.4}$$

By imposing quantitative control on the increasing roughness of W^ϵ for small ϵ, we are able to prove the following: