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Kantorovich potentials and continuity of total cost for relativistic cost functions

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A B S T R A C T

In this paper we consider the optimal mass transport problem for relativistic cost functions, introduced in [12] as a generalization of the relativistic heat cost. A typical example of such a cost function is $c_t(x,y) = h(\frac{y-x}{t})$, h being a strictly convex function when the variable lies on a given ball, and infinite otherwise. It has been already proved that, for every t larger than some critical time $T > 0$, existence and uniqueness of optimal maps hold; nonetheless, the existence of a Kantorovich potential is known only under quite restrictive assumptions. Moreover, the total cost corresponding to time t has been only proved to be a decreasing right-continuous function of t. In this paper, we extend the existence of Kantorovich potentials to a much broader setting, and we show that the total cost is a continuous function. To obtain both results the two main crucial steps are a refined "chain lemma" and the result that, for $t > T$, the points moving at maximal distance are negligible for the optimal plan.

R É S U M É

Nous considérons dans ce papier le problème de transport optimal pour des fonctions de coût relativistes, généralisation du coût correspondant à une propagation relativiste de la chaleur. Un exemple typique d'une telle fonction coût s'écrit $c(t,x) = h(\frac{y-x}{t})$, h étant une fonction strictement convexe définie sur une boule et infinie en dehors de cette boule. L'existence d'une unique application de transport pour tout t supérieur à un temps critique $T > 0$ est déja connue mais l'obtention des potentiels de Kantorovich correspondants était prouvée sous des hypothèses restrictives. Dans cet article, nous étendons l'existence de potentiels de Kantorovich à un cadre bien plus large et nous montrons que le coût total est une fonction continue. Les deux étapes cruciales pour obtenir ces résultats sont l'obtention d'un

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"chain lemma" élaboré et la preuve du fait que l'ensemble des points déplacés à distance maximale est négligeable.

1. Introduction

In this paper, we consider the classical mass transport problem with a particular class of cost functions c. While the case when c is strictly convex and real-valued is well understood (we refer to [34] for a survey), our goal is to continue the study of a wide class of non-real-valued cost functions, called "relativistic costs", which were introduced in [12] as a generalization to the "relativistic heat cost".

More precisely, the *relativistic heat cost*, introduced by Brenier in [14], is defined as $c(x,y) = h(y-x)$, where h is given by the formula

$$h(z) = \begin{cases} 1 - \sqrt{1-|z|^2} & |z| \leq 1\,, \\ +\infty & |z| > 1\,. \end{cases} \tag{1.1}$$

As explained by Brenier in the aforementioned paper, this cost function can be used in order to study a relativistic heat equation, where the region where the cost is infinite comes from the fact that the heat has finite speed propagation. Afterwards, the corresponding relativistic heat equation has also been studied by McCann and Puel in [28] by means of the Jordan–Kinderlehrer–Otto approach introduced in [25], and by Andreu, Caselles and Mazón via PDE methods in a series of papers [3,5,4,6–9,16]. We refer to [14,12] and the references therein for more on this topic.

Notice that, from the mathematical point of view, a quite interesting feature of this cost function is that it is strictly convex and bounded on its domain, hence in particular it is not continuous on $\mathbb{R}^n \times \mathbb{R}^n$ (while the non-real-valued cost functions have been often assumed to be continuous, see for instance [23,22,29,30, 33,11]).

More in general, one can consider costs "of relativistic type", meaning that they are strictly convex and bounded in a strictly convex domain $\mathscr{C}$ and $+\infty$ outside. The corresponding optimal mass transport problem has been studied in [12], see also [24].

In the classical case (strictly convex and real-valued cost function), a lot can be said; in particular, an optimal transport map exists, is unique, and it is induced by a *Kantorovich potential* φ (for instance the optimal transport map is simply the gradient of φ when the cost is the Euclidean norm squared, see [27, 13]). This potential is also related to the so-called Kantorovich dual problem. For non-negative and lower semi-continuous cost functions, the following duality result is well known (see [21]).

Theorem 1.1 *(Kantorovich duality). Let μ and ν be two probability measures with finite second order moment on $\mathbb{R}^n$. Then, the following equality holds:*

$$\sup_{\mathcal{A}} \left\{ \int_{\mathbb{R}^n} \varphi(x) d\mu(x) + \int_{\mathbb{R}^n} \psi(y) d\nu(y) \right\} = \min_{\gamma \in \Pi(\mu,\nu)} \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x,y)\, d\gamma(x,y) < +\infty.$$

Here and in the following, $\Pi(\mu,\nu)$ denotes the set of positive measures on $\mathbb{R}^n \times \mathbb{R}^n$ whose first (resp., second) marginal is μ (resp., ν), while $\mathcal{A}$ is the set of pairs (φ, ψ) of Lipschitz functions defined on $\mathbb{R}^n$ that satisfy $\varphi(x) + \psi(y) \leq c(x,y)$ for all $x, y \in \mathbb{R}^n$.

However, it should be emphasized that in general, Kantorovich's dual problem (on the left hand side) has no solutions, counterexamples can be found for instance in [10]. A sufficient condition for maximisers to exist is that the cost function is a bounded and uniformly continuous function, see [34]; a more general criterion

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