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Smoothing and non-smoothing via a flow tangent to the Ricci flow

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ABSTRACT

We study a transformation of metric measure spaces introduced by Gigli and Mantegazza consisting in replacing the original distance with the length distance induced by the transport distance between heat kernel measures. We study the smoothing effect of this procedure in two important examples. Firstly, we show that in the case of particular Euclidean cones, a singularity persists at the apex. Secondly, we generalize the construction to a sub-Riemannian manifold, namely the Heisenberg group, and show that it regularizes the space instantaneously to a smooth Riemannian manifold.

1. Introduction

There are many ways to deform a Riemannian manifold into a singular metric space as discussed for instance in the influential essay of Gromov [1]. We are interested in the opposite question whether there exists a deformation, intrinsically defined for a wide class of metric spaces that instantaneously turns the space into a Riemannian manifold. In this paper, we investigate a method that has been introduced by Gigli and Mantegazza [2]. We examine its regularization properties in two important cases: Euclidean cones and the Heisenberg group. These are emblematic examples of Alexandrov spaces and sub-Riemannian spaces respectively. We also discuss normed vector spaces where the transformation turns out to be the identity as an example of Finsler structures.

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Before we state our results we briefly explain the main features of the construction of Gigli and Mantegazza which is based on the interplay of optimal transport and Ricci curvature. The starting point is a metric measure space (X, d, m) on which a reasonable notion of heat kernel can be defined. For $t > 0$ a new distance $d_t(x, y)$ is defined as the length distance induced by the L^2-Wasserstein distance built from d between the heat kernel measures centered at x and y.

The striking feature of this approach is the following main result of [2]: when (X, d, m) is a Riemannian manifold then d_t is induced by a smooth metric tensor g_t that is tangent to the Ricci flow, i.e. $\partial_t|_{t=0} g_t = -2\,\mathrm{Ric}$ in a weak sense. Gigli and Mantegazza then generalize this construction to metric measure spaces satisfying generalized Ricci curvature lower bounds, more precisely the RCD condition, which ensures existence of a well-behaved heat kernel. This can be seen as a first step towards constructing a Ricci flow for non-smooth initial data. Synthetic characterizations of super-Ricci flows based on optimal transport have been obtained by McCann and Topping [3] and by Sturm [4].

Since d_t can be thought of as a sort of convolution of the original distance with the heat kernel, having the smoothing effect of the heat equation and Ricci flow in mind, one might expect that this procedure gives a canonical way of regularizing the metric measure space.

A first study of the regularizing effects of the Gigli–Mantegazza flow has been performed by Bandara, Lakzian and Munn [5] in the case where the distance d is induced by a metric tensor with low regularity and isolated conic singularities. It is shown that d_t is induced by a metric tensor with at least the same regularity away from the original singular set. The question, what happens at the singularities has been left unanswered.

1.1. Non-smoothing of cones

In the present paper, we give an answer showing that conic singularities can persist under the Gigli–Mantegazza transformation. We analyze in detail the transformation for two specific Euclidean cones of angle π and $\pi/2$. Our results are the following (see Theorem 3.11 and Proposition 3.10 below).

Theorem 1.1. *Let $C(\pi)$ be the two-dimensional Euclidean cone of angle π and d its distance. For every $t > 0$ the convoluted distance d_t has a conic singularity of angle $\sqrt{2}\pi$ at the apex.*

As t goes to zero, the metric space $(C(\pi), d_t)$ tends to $(C(\pi), d)$ pointwise and in the pointed Gromov–Hausdorff topology. As t goes to infinity, it tends to the Euclidean cone of angle $\sqrt{2}\pi$ in the pointed Gromov–Hausdorff topology.

In fact, it turns out that for fixed $\theta > 0$ all spaces $(C(\theta), d_t)$ for $t > 0$ are isometric up to a multiplicative constant. An isometry is induced by the radial dilation $x \in C(\theta) \mapsto t^{-1/2}x$. Our second result shows that for the cone of angle $\pi/2$ the behavior of the singularity is even worse (see Theorem 3.17 and Proposition 3.16 below).

Theorem 1.2. *Let $C(\pi/2)$ be the two-dimensional Euclidean cone of angle $\pi/2$ and d its distance. For every $t > 0$, the distance d_t has a cusp singularity at the apex, more precisely the asymptotic angle is zero.*

As t goes to zero, the metric space $(C(\pi/2), d_t)$ tends to $(C(\pi/2), d)$ pointwise and in the pointed Gromov–Hausdorff topology. As t goes to infinity, it tends to $\mathbb{R}^+$ with the Euclidean distance in the pointed Gromov–Hausdorff sense.

The reason why we focus on these two specific cones is that they can be conveniently represented as quotients of $\mathbb{R}^2$ under rotation by π and $\pi/2$ respectively. It turns out that the convoluted distance d_t is the length distance induced by the L^2-Wasserstein distance between a mixtures of two (respectively four) rotated copies of Gaussian measures with variance $2t$.

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