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Nonhomogeneous boundary-value problems for one-dimensional nonlinear Schrödinger equations

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ABSTRACT

This paper is concerned with initial-boundary-value problems (IBVPs) for a class of nonlinear Schrödinger equations posed either on a half line $\mathbb{R}^+$ or on a bounded interval $(0, L)$ with nonhomogeneous boundary conditions. For any s with $0 \leq s < 5/2$ and $s \neq 1/2$, it is shown that the relevant IBVPs are locally well-posed if the initial data lie in the L^2-based Sobolev spaces $H^s(\mathbb{R}^+)$ in the case of the half line and in $H^s(0, L)$ on a bounded interval, provided the boundary data are selected from $H_{loc}^{(2s+1)/4}(\mathbb{R}^+)$ and $H_{loc}^{(s+1)/2}(\mathbb{R}^+)$, respectively. (For $s > \frac{1}{2}$, compatibility between the initial and boundary conditions is also needed.) Global well-posedness is also discussed when $s \geq 1$. From the point of view of the well-posedness theory, the results obtained reveal a significant difference between the IBVP posed on $\mathbb{R}^+$ and the IBVP posed on $(0, L)$. The former is reminiscent of the theory for the pure initial-value problem (IVP) for these Schrödinger equations posed on the whole line $\mathbb{R}$ while the theory on a bounded interval looks more like that of the pure IVP posed on a periodic domain. In particular, the regularity demanded of the boundary data for the IBVP on $\mathbb{R}^+$ is consistent with the temporal trace results that obtain for solutions of the pure IVP on $\mathbb{R}$, while the slightly higher regularity of boundary data for the IBVP on $(0, L)$ resembles what is found for temporal traces of spatially periodic solutions.

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RÉSUMÉ

L'article est consacré à l'étude de problèmes avec conditions initiales et aux limites pour une classe d'équations de Schrödinger non linéaires considérées soit sur la demi-droite $\mathbb{R}^+$, soit sur un intervalle borné $]0, L[$, avec des conditions aux limites non homogènes. Pour tout s vérifiant $0 \leq s < 5/2$ et $s \neq 1/2$, on montre que le problème avec conditions initiales et aux limites est localement bien posé si la donnée initiale est dans l'espace de Sobolev $H^s(\mathbb{R}^+)$ dans le cas de la demi-droite, et dans $H^s(0, L)$ dans le cas de l'intervalle borné, pourvu que les données aux limites soient choisies dans $H_{loc}^{(2s+1)/4}(\mathbb{R}^+)$ et $H_{loc}^{(s+1)/2}(\mathbb{R}^+)$, respectivement. Pour $s > \frac{1}{2}$, les données initiales et les données aux limites doivent être compatibles. Le

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caractère globalement bien posé est également discuté quand $s \geq 1$. Du point de vue de la théorie des problèmes bien posés, les résultats obtenus ici révèlent une différence significative entre les problèmes considérés sur $\mathbb{R}^+$ et ceux sur $]0, L[$. La théorie sur la demi-droite fait penser à celle développée pour les problèmes avec données initiales sur la droite, tandis que celle sur un intervalle borné s'apparente à celle développée pour les problèmes sur un domaine périodique. En particulier, la régularité requise pour les données aux limites dans le cas de la demi-droite est cohérente avec les résultats de régularité des traces temporelles des solutions d'un problème avec donnée initiale sur $\mathbb{R}$, tandis que la régularité légèrement supérieure requise pour les données aux limites dans le cas de l'intervalle $]0, L[$ ressemble à celle obtenue pour les traces temporelles des solutions périodiques en espace.

1. Introduction

Studied here are initial-boundary-value-problems (IBVPs) for nonlinear Schrödinger equations posed either on a half line $\mathbb{R}^+$, *viz.*

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2} u = 0, & x \in \mathbb{R}^+, \ t \in \mathbb{R}, \\ u(x,0) = \phi(x), & u(0,t) = h(t), \end{cases} \tag{1.1}$$

or on a finite interval $(0, L)$,

$$\begin{cases} iu_t + u_{xx} + \lambda |u|^{p-2} u = 0, & x \in (0,L), \quad t \in \mathbb{R}, \\ u(x,0) = \phi(x), \qquad u(0,t) = h_1(t), & u(L,t) = h_2(t). \end{cases} \tag{1.2}$$

Here, the parameter λ is a non-zero real number and $p \geq 3$.[1] Note that, due to the symmetry of the equation with respect to the change of variables $x \to -x$, results established for (1.1) carry over *mutatis muntandis* to the quarter-plane problem where $\mathbb{R}^+$ is replaced by $\mathbb{R}^-$. (The situation regarding the quarter-plane problems posed on $\mathbb{R}^+$ and $\mathbb{R}^-$ for the Korteweg–de Vries equation are significantly different on the other hand.) In all cases where (1.1) and (1.2) arise in practice, the second-order derivative models dispersive effects, which is to say the tendency of waves to spread out due to the fact that different wavelengths propagate with different speeds, while the $|u|^{p-2}u$-term accounts for a variety of nonlinear effects.

Nonlinear Schrödinger equations are derived as models for a considerable range of applications. This includes propagation of light in fiber optics cables, certain types of shallow and deep surface water waves, Langmuir waves in a hot plasma and, in more general forms, in Bose–Einstein condensate theory. In the case of gravity waves on the surface of an inviscid liquid, the parameter λ depends upon the undisturbed depth of the water, becoming negative in water deep with respect to the wavelength of the wavetrain. A particularly interesting application of nonlinear Schrödinger (NLS henceforth) equations has been their use in attempting to explain the somewhat mysterious formation of rogue waves in the ocean and in optical propagation (see [6,7,24,41]).

In many of the physical applications mentioned above, the independent variable x is a coordinate representing position in the medium of propagation, t is proportional to elapsed time and $u(x,t)$ is a velocity or an amplitude at the point x at time t. One configuration that arises naturally in making predictions of waves in water is to take $x \in \mathbb{R}^+ = \{x \,|\, x \geq 0\}$ and specify $u(0,t)$ for $t > 0$. This corresponds to a given

[1] Only the case $p \geq 3$ is considered here, but a substantial part of the theory goes through under the weaker hypothesis $p > 2$.