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Global classical solutions to partially dissipative hyperbolic systems violating the Kawashima condition

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ABSTRACT

This paper considers the Cauchy problem for the quasilinear hyperbolic system of balance laws in $\mathbb{R}^d$, $d \geq 2$. The system is partially dissipative in the sense that there is an eigen-family violating the Kawashima condition. By imposing certain supplementary degeneracy conditions with respect to the non-dissipative eigen-family, global unique smooth solutions near constant equilibria are constructed. The proof is based on the introduction of the partially normalized coordinates, a delicate structural analysis, a family of scaled energy estimates with minimum fractional derivative counts and a refined decay estimates of the dissipative components of the solution.

RÉSUMÉ

Dans cet article, on considère le problème de Cauchy pour le système hyperbolique quasi linéaire des lois de bilan dans $\mathbb{R}^d$, $d \geq 2$. Le système est partiellement dissipatif, dans le sens où il y a une famille de fonctions propres qui ne respectent pas la condition de Kawashima. En imposant certaines conditions de dégénérescence supplémentaires par rapport à la famille de fonctions propres non dissipatives, on construit les solutions globales, lisses et uniques, près des équilibres constants. La démonstration utilise l'introduction des coordonnées partiellement normalisées, une analyse structurale délicate, une famille d'estimations d'énergie mise à l'échelle avec des comptes minimaux de dérivées fractionnaires, et une estimation précise du retard pour les composantes dissipatives des solutions.

1. Introduction

Consider the following n-component quasilinear hyperbolic system of balance laws in d space dimensions:

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$$\begin{cases} \partial_t \left(G(u)\right) + \sum_{k=1}^{d} \left(F^k(u)\right)_{x_k} = S(u), \\ u \mid_{t=0} = u_0. \end{cases} \tag{1}$$

Here $u = (u_1, \dots, u_n)^T : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^n$ is the unknown, $G = (G_1, \dots, G_n)^T : \mathbb{R}^n \to \mathbb{R}^n$, $F^k = (F_1^k, \dots, F_n^k)^T : \mathbb{R}^n \to \mathbb{R}^n$ $(k = 1, \dots, d)$ and $S = (S_1, \dots, S_n)^T : \mathbb{R}^n \to \mathbb{R}^n$ are given smooth functions. It is assumed that $\nabla_u G(u)$ is invertible, hence, by the chain rule, the system can be rewritten into

$$\begin{cases} \partial_t u + \sum_{k=1}^{d} A^k(u) u_{x_k} = Q(u), \\ u \mid_{t=0} = u_0. \end{cases} \tag{2}$$

Here the coefficient matrices $A^k(u) = (\nabla_u G(u))^{-1} \nabla_u F^k(u)$ $(k = 1, \dots, d)$ and the inhomogeneous term $Q(u) = (\nabla_u G(u))^{-1} S(u)$. By hyperbolicity, for any $u \in \mathbb{R}^n$ and any $\omega = (\omega_1, \dots, \omega_d)^T \in \mathbb{S}^{d-1}$ the matrix

$$A(u, \omega) := \sum_{k=1}^{d} \omega_k A^k(u) \tag{3}$$

has n real eigenvalues $\lambda_1(u, \omega), \dots, \lambda_n(u, \omega)$ and a complete set of left (resp. right) eigenvectors $l_i(u, \omega) = (l_{i1}(u, \omega), \dots, l_{in}(u, \omega))$ (resp. $r_i(u, \omega) = (r_{1i}(u, \omega), \dots, r_{ni}(u, \omega))^T$) $(i = 1, \dots, n)$. It is assumed that $\lambda_i(u, \omega)\,(i = 1, \dots, n)$ are smooth with respect to u and ω, while $l_i(u, \omega)$, $r_i(u, \omega)\,(i = 1, \dots, n)$ are all smooth functions of u for any given $\omega \in \mathbb{S}^{d-1}$. One may normalize the eigenvectors so that

$$l_i(u, \omega) r_{i'}(u, \omega) \equiv \delta_{ii'}, \quad i, i' = 1, \dots, n. \tag{4}$$

It is well known that for any given suitably smooth initial data u_0, the system (1) has a unique local smooth solution [11,19], but in general the solution develops singularities in finite time even for small and smooth initial data [4,16,19].

It has been known that there are two structures that can prevent the finite-time singularity formation. One of them is the (weak) linear degeneracy for the homogeneous systems, which was firstly introduced in 1D study (see [3] and [16]). The other one is the dissipation effect induced by inhomogeneous terms. A well-known assumption in this aspect is given by the strict dissipation, which requires a damping term to enter into each of the equations of the system, see [16]. Notice that for some physical problems, this condition is too strong, while in many applications and as the interest in this paper $Q(u)$ takes the form

$$Q(u) = (0, \dots, 0, Q_{r+1}(u), \dots, Q_n(u))^T, \tag{5}$$

which seems to produces the dissipation effect only in the last $n - r$ equations. In this case, the system (1) is referred to as a partially dissipative hyperbolic system in many literatures, such as [1,2,7,26].

A constant vector $u_e \in \mathbb{R}^n$ is called an equilibrium state of the system (1) if $S(u_e) = 0$, or equivalently, $Q(u_e) = 0$. The main subject of this paper is to search for suitable structural conditions, under which (1) has a unique global smooth solution near u_e. Without loss of generality, one may take $u_e = 0$, i.e. $Q(0) = 0$. One may also suppose that $G(0) = 0$ and $\nabla_u G(0) = I_n$. The first essential assumption on the structure of the system is the following entropy dissipation condition:

($\mathfrak{A}1$) The matrix $\left(\frac{\partial Q_p}{\partial u_{p'}}(0)\right)_{p,p'=r+1}^{n}$ is invertible.

($\mathfrak{A}2$) There exist a strictly convex smooth entropy function $\bar\eta(G)$ and d smooth entropy flux functions $\bar\psi^k(G)$ $(k = 1, \dots, d)$ for the system (1) in terms of G, such that for all G,