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A Jellett type theorem for the Levi curvature

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A B S T R A C T

In this paper we prove a Jellett-type theorem for real hypersurfaces in $\mathbb{C}^2$ with respect to the Levi curvature. We provide as applications rigidity results for domains with circular symmetries.

R É S U M É

Dans cet article, on démontre un théorème de type Jellett pour des hypersurfaces réelles de $\mathbb{C}^2$ par rapport à la courbure de Levi. Comme application, on fournit des résultats de rigidité pour des domaines avec symétries circulaires.

1. Introduction and state of the art

Let M be a smooth orientable hypersurface in $\mathbb{R}^{n+1}$, with $n \geq 1$. We say that M is starshaped if it is a boundary of a bounded domain which is strictly starshaped. In 1853 Jellett proved the following:

Theorem *(Jellett, [16]). Any starshaped hypersurface M in $\mathbb{R}^{n+1}$ with constant mean curvature is a sphere.*

Jellett proved his theorem for two-dimensional surfaces in $\mathbb{R}^3$, but the same techniques work in any dimension.

Remark 1.1. In the case $n = 1$, the Jellett method works without the starshapedness assumption. In fact, a closed embedded curve in $\mathbb{R}^2$ whose curvature is strictly positive bounds a convex domain, which is in particular starshaped with respect to any interior point.

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As it is very well-known, a fundamental result by Aleksandrov [1] says that the assumption of being starshaped is not needed in any dimension.

About Jellett's theorem, which dates nearly a century before Aleksandrov's, we have to mention that almost 50 years later (in 1899) Liebmann proved a weaker result that turned out to have more resonance in the future literature: he proved in [22] that the only closed convex hypersurfaces in $\mathbb{R}^3$ with constant mean curvature are the spheres. A genuine generalization of Jellett's theorem was established in 1951 by Hopf, who proved that the only compact contractible surfaces in $\mathbb{R}^3$ with constant mean curvature are the spheres (see [11]): it is worth to notice that the technique of Hopf effectively works for two-dimensional surfaces.

Just few years later, Aleksandrov proved in [1] that the spheres are the only closed hypersurfaces with constant mean curvature by introducing his celebrated moving planes technique. Later on, an alternative proof of the same result was given by Reilly in [36], where he adopted a new integral approach. We will come back again on these two different approaches.

Now, let us briefly sketch the proof of Jellett result for the reader's convenience (see also [23, Chapter 2]). Suppose that $M \subset \mathbb{R}^{n+1}$ is starshaped with respect to the origin. Let ν be the unit outward normal to M, and $p \in \mathbb{R}^{n+1}$ be the position vector. Let also $\langle \cdot, \cdot \rangle$ and $|\cdot|$ be the usual inner product and norm in $\mathbb{R}^{n+1}$, then we denote by $\psi(p) = \frac{|p|^2}{2}$ and $\lambda(p) = \langle p, \nu \rangle$ respectively the square of the distance and the support function. A straightforward computation shows that

$$\Delta_M \psi = n - nH\lambda,$$

where Δ_M and H stand respectively for the Laplace–Beltrami operator on M and for its mean curvature. In our notations nH is the trace of the second fundamental form h. Moreover, if H is constant, by the Codazzi equations we get

$$\Delta_M \lambda = nH - \|h\|^2 \lambda.$$

Here we used the notation $\|\cdot\|^2$ for the squared norm of a matrix, namely the sum of all of its squared coefficients. We recall that, for any symmetric $n \times n$ matrix Q, it holds true that

$$\|Q\|^2 \geq \frac{1}{n}(\mathrm{trace}(Q))^2, \tag{1}$$

and the equality occurs if and only if the matrix Q is a multiple of the identity. This fact and the starshapedness assumption imply

$$\Delta_M (H\psi - \lambda) = (\|h\|^2 - nH^2)\lambda \geq 0. \tag{2}$$

The function $H\psi - \lambda$ is then Δ_M-subharmonic, and thus constant being M closed (compact, without boundary). In particular, since λ is strictly positive, $\|h\|^2 = nH^2 = \frac{1}{n}(\mathrm{trace}(h))^2$. The equality case in (1) says that M is umbilical, therefore it must be a sphere.

Here we want to follow a similar strategy for the case of a threedimensional hypersurface M in $\mathbb{C}^2$ and to prove a theorem 'à la Jellett' when we consider the Levi mean curvature in place of the standard mean curvature. The Levi curvature is a sort of degenerate-elliptic analogue of the classical mean curvature: it was introduced and studied in [2,10]. Roughly speaking, one considers the restriction of the second fundamental form to the holomorphic tangent space (see Section 2 for the precise definitions). Such restriction involves a lack of information and hence a lack of ellipticity in the relative operator. However, in the case of a suitable non-flatness, the direction of missing ellipticity is recovered through bracket commutations: the Levi operator can be thus seen as a degenerate-elliptic operator of sub-Riemannian type. This very special feature has been successfully exploited, e.g. by Citti–Lanconelli–Montanari in [9] where they were able to prove a regularity

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