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Optimal asymptotic behavior of the vorticity of a viscous flow past a two-dimensional body

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The asymptotic behavior of the vorticity for the steady incompressible Navier–Stokes equations in a two-dimensional exterior domain is described in the case where the velocity at infinity $\boldsymbol{u}_\infty$ is nonzero. It is well known that the asymptotic behavior of the velocity field is given by the fundamental solution of the Oseen system which is the linearization of the Navier–Stokes equation around $\boldsymbol{u}_\infty$. Concerning the vorticity, the previous asymptotic expansions were relevant only inside a parabolic region called the wake region. Here we present an asymptotic expansion for the vorticity relevant everywhere. Surprisingly, the found asymptotic behavior is not given by the Oseen linearization and has a power of decay that depends on the data. This strange behavior is specific to the two-dimensional problem and is not present in three dimensions.

R É S U M É

Le comportement asymptotique de la vorticité pour les équations de Navier–Stokes incompressibles dans un domaine extérieur bidimensionnel est décrit dans le cas où la vitesse à l'infini $\boldsymbol{u}_\infty$ est non nulle. Il est bien connu que le développement asymptotique du champ de vitesses est donné par la solution fondamentale de l'équation d'Oseen, qui correspond à la linéarisation des équations de Navier–Stokes autour de $\boldsymbol{u}_\infty$. Concernant la vorticité, les développements asymptotiques antérieurs étaient pertinents seulement dans la région parabolique du sillage. Dans cet article, on présente un développement asymptotique pour la vorticité pertinent partout. De manière surprenante, le comportement asymptotique n'est pas donné par la linéarisation d'Oseen et possède une décroissance dépendant des données. Cette particularité est spécifique au problème bidimensionnel et n'est pas présente en trois dimensions.

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1. Introduction

The stationary flow of an incompressible fluid past a body is described by the Navier–Stokes equations,

$$\begin{aligned} \Delta \boldsymbol{u} - \boldsymbol{\nabla} p &= \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{f} \,, & \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \,, \\ \boldsymbol{u}|_{\partial\Omega} &= \boldsymbol{u}^* \,, & \lim_{|\boldsymbol{x}|\to\infty} \boldsymbol{u} &= \boldsymbol{u}_\infty \,, \end{aligned} \tag{1a}$$

in the exterior domain $\Omega = \mathbb{R}^2 \setminus \bar{B}$, where B is a bounded simply connected Lipschitz domain, $\boldsymbol{f}$ a source force of compact support in Ω, $\boldsymbol{u}_\infty \neq \boldsymbol{0}$ the velocity at infinity and $\boldsymbol{u}^*$ any boundary condition with no net flux,

$$\int_{\partial\Omega} \boldsymbol{u}^* \cdot \boldsymbol{n} = 0 \,. \tag{1b}$$

In view of the symmetries of the equation, we assume without loss of generality that $\boldsymbol{u}_\infty = 2\boldsymbol{e}_1$.

This system has been subject to many investigations under different smoothness hypotheses on the domain, on $\boldsymbol{f}$ and on $\boldsymbol{u}^*$, see Galdi [1, Chapter XII] for a complete statement of the main results known for this problem. Leray [2] has shown the existence of a weak solution $\boldsymbol{u} \in \dot{H}^1(\Omega)$, but with the procedure he used, he was unable to verify that $\boldsymbol{u}$ tends to $\boldsymbol{u}_\infty$ at large distances. Gilbarg and Weinberger [3,4] have shown that any Leray solution $\boldsymbol{u}$ either converges at large distances in a weak sense to some constant vector $\boldsymbol{u}_0$ or diverges in the same weak sense. Later on, Amick [5] proved that if $\boldsymbol{f} = \boldsymbol{0}$ and $\boldsymbol{u}^* = \boldsymbol{0}$, then $\boldsymbol{u} \in L^\infty(\Omega)$ and therefore $\boldsymbol{u}$ converges to a constant $\boldsymbol{u}_0$ at infinity. However, the question if $\boldsymbol{u}_0 = \boldsymbol{u}_\infty$ is still open in general.

The convergence of weak solutions to $\boldsymbol{u}_\infty$ at large distances being unknown, we consider the subspace of physically reasonable solutions. We recall that a solution $\boldsymbol{u} \in \dot{H}^1(\Omega)$ is physically reasonable in the sense of Smith [6, §4] if $\boldsymbol{u} - \boldsymbol{u}_\infty = O(r^{-1/4-\varepsilon})$ for some $\varepsilon > 0$. Finn and Smith [7], Galdi [8,9,1] used the Oseen approximation and a fixed point technique to prove existence and uniqueness of physically reasonable solutions to (1) under smallness and regularity assumptions on B, $\boldsymbol{f}$, and $\boldsymbol{u}^*$. We note that without smallness assumptions such an existence result is still open.

The asymptotic structure of the physically reasonable solutions was presented by Babenko [10] who shows in particular that the velocity behaves at infinity like the Oseen fundamental solution. The asymptotic expansion of the velocity was also given under more general assumptions by Galdi and Sohr [11], Sazonov [12]. The asymptotic behavior of the vorticity was first given by Babenko [10, Theorem 8.1] but only in the wake region, and then by Clark [13, Theorem 3.5']. These two results are relevant only in the wake region, *i.e.* for $|\boldsymbol{x}| - x_1 \leq 1$, because otherwise, the remainder decays slower than the asymptotic term which is given by the Oseen linearization. Here we prove that the asymptote which is also valid outside the wake region is in fact not given by the Oseen linearization. Our main result states that the asymptote of the vorticity $\omega = \boldsymbol{\nabla} \wedge \boldsymbol{u}$ of a physically reasonable solution is given in polar coordinates (r, θ) by

$$\omega(\boldsymbol{x}) = r^{A_1(1-\cos\theta) - A_2 \sin\theta} \left[\frac{\mu(\theta)}{r^{1/2}} + O\left(\frac{1}{r^{1/2+\varepsilon}} \right) \right] \mathrm{e}^{-r(1-\cos\theta)} \,,$$

as $r \to \infty$ for all $\varepsilon \in \left(0, \frac{1}{4}\right)$, where $\boldsymbol{A} = (A_1, A_2) \in \mathbb{R}^2$ is a vector depending linearly on the net force

$$\boldsymbol{F} = \int_{\Omega} \boldsymbol{f} - \int_{\partial\Omega} \left(\mathbf{T}(\boldsymbol{u}, p) - \boldsymbol{u} \otimes \boldsymbol{u} \right) \cdot \boldsymbol{n} \,, \qquad \mathbf{T}(\boldsymbol{u}, p) = \boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^{\mathrm{T}} - p\mathbf{1} \,, \tag{2}$$

and μ is a 2π-periodic continuous function depending on $\boldsymbol{f}$ and $\boldsymbol{u}^*$ (see Theorem 2 for the precise formulation). This result is quite astonishing, because the decay rate of the vorticity depends on the net force $\boldsymbol{F}$

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