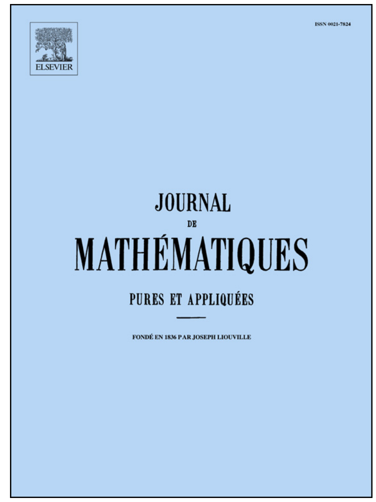

Abelian tensors

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ABELIAN TENSORS

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ABSTRACT. We analyze tensors in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfying Strassen's equations for border rank m. Results include: two purely geometric characterizations of the Coppersmith-Winograd tensor, a reduction to the study of symmetric tensors under a mild genericity hypothesis, and numerous additional equations and examples. This study is closely connected to the study of the variety of m-dimensional abelian subspaces of $\mathrm{End}(\mathbb{C}^m)$ and the subvariety consisting of the Zariski closure of the variety of maximal tori, called the variety of reductions.

Sommaire. Nous étudions des tenseurs dans $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ satisfaisant les équations de Strassen lorsque le rang du bord vaut m. Les résultats obtenus comprennent : deux caractérisations purement géométriques du tenseur de Coppersmith-Winograd, une réduction à l'étude des tenseurs symétriques sous une hypothèse raisonnable de généricité, et beaucoup de nouveaux exemples et équations. Cette étude est liée de près à l'étude de la variété des sous-espaces abéliens de dimension m de $\mathrm{End}(\mathbb{C}^m)$ et la sous-variété obtenue comme l'adhérence de Zariski de la variété des tores maximaux, appelée variété des réductions.

1. INTRODUCTION

The rank and border rank of a tensor $T \in \mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (defined below) are basic measures of its complexity. Central problems are to develop techniques to determine them (see, e.g., [27, 13, 15, 23]). Complete resolutions of these problems are currently out of reach. For example, neither problem is solved already in $\mathbb{C}^4 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$. This article focuses on a very special class of tensors, those satisfying Strassen's commutativity equations (see §2.1). The study of such tensors is related to the classical problem of studying spaces of commuting matrices, see, e.g. [21, 44, 22, 26].

To completely understand border rank, it would be sufficient to understand the case of border rank m in $\mathbb{C}^m \otimes \mathbb{C}^m \otimes \mathbb{C}^m$ (see [27, Cor. 7.4.1.2]). We study this problem under two genericity hypotheses - *concision*, which essentially says we restrict to tensors that are not contained in some $\mathbb{C}^{m-1} \otimes \mathbb{C}^m \otimes \mathbb{C}^m$, and 1_A-*genericity*, which is defined below. Even under these genericity hypotheses, the problem is still subtle.

Let A, B, C be complex vector spaces of dimensions $\mathbf{a}, \mathbf{b}, \mathbf{c}$, let $T \in A \otimes B \otimes C$ be a tensor. (In bases T is a three dimensional matrix of size $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$.) We may view T as a linear map $T : A^* \to B \otimes C \simeq \mathrm{Hom}(C^*, B)$. (In bases, $T((\alpha_1, \cdots, \alpha_{\mathbf{a}}))$ is the $\mathbf{b} \times \mathbf{c}$ matrix α_1 times the first slice of the $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ matrix, plus α_2 times the second slice ... plus $\alpha_{\mathbf{a}}$ times the $\mathbf{a}$-th slice.) One may recover T up to isomorphism from the space of linear maps $T(A^*)$.

One says T has *rank one* if $T = a \otimes b \otimes c$ for some $a \in A$, $b \in B$ and $c \in C$, and the *rank* of T, denoted $\mathbf{R}(T)$ is the smallest r such that T may be expressed as the sum of r rank one tensors. Rank is not semi-continuous, so one defines the *border rank* of T, denoted $\underline{\mathbf{R}}(T)$, to be the smallest r such that T is a limit of tensors of rank r, or equivalently (see e.g. [27, Cor. 5.1.1.5]) the smallest r such that T lies in the Zariski closure of the set of tensors of rank r.

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