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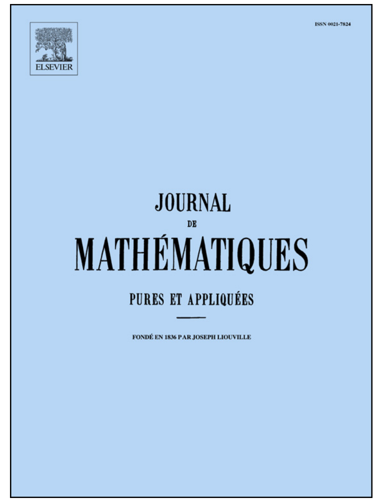

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The Klein-Gordon Equation on $\mathbb{Z}^2$ and the Quantum Harmonic Lattice

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Abstract

The discrete Klein-Gordon equation on a two-dimensional square lattice satisfies an $\ell^1 \mapsto \ell^\infty$ dispersive bound with polynomial decay rate $|t|^{-3/4}$. We determine the shape of the light cone for any choice of the mass parameter and relative propagation speeds along the two coordinate axes. Fundamental solutions experience the least dispersion along four lines interior to the light cone rather than along its boundary, and decay exponentially to arbitrary order outside the light cone. The overall geometry of the propagation pattern and its associated dispersive bounds are independent of the particular choice of parameters. In particular there is no bifurcation of the number or type of caustics that are present. The dispersive bounds imply global well-posedness for small solutions of a nonlinear discrete Klein-Gordon equation.

The discrete Klein-Gordon equation is a classical analogue of the quantum harmonic lattice. In the quantum setting, commutators of time-shifted observables experience the same decay rates as the corresponding Klein-Gordon solutions, which depend in turn on the relative location of the observables' support sets.

1. Introduction

The wave equation $u_{tt} - \Delta u = 0$ on $\mathbb{R}^{2+1}$ is explicitly solved via Poisson's formula, in which initial data $u(x,0) = g(x)$, $u_t(x,0) = h(x)$ determines the unique solution

$$u(x,t) = \frac{\operatorname{sign}(t)}{2\pi} \int_{|y-x|<|t|} \frac{h(y) + \frac{1}{t}(g(y) + \nabla g(y) \cdot (y-x))}{\sqrt{t^2 - |y-x|^2}}\, dy$$

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