Applied Numerical Mathematics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

The lumped mass FEM for a time-fractional cable equation

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ARTICLE INFO

Article history:
Received 8 November 2017
Received in revised form 18 February 2018
Accepted 21 May 2018
Available online xxxx

Keywords:
Time-fractional cable equation
Lumped mass FEM
Laplace transform
Convolution quadrature
Error estimate
Nonsmooth data

ABSTRACT

We consider the numerical approximation of a time-fractional cable equation involving two Riemann–Liouville fractional derivatives. We investigate a semidiscrete scheme based on the lumped mass Galerkin finite element method (FEM), using piecewise linear functions. We establish optimal error estimates for smooth and middly smooth initial data, i.e., $v \in H^q(\Omega) \cap H_0^1(\Omega)$, $q = 1, 2$. For nonsmooth initial data, i.e., $v \in L^2(\Omega)$, the optimal $L^2(\Omega)$-norm error estimate requires an additional assumption on mesh, which is known to be satisfied for symmetric meshes. A quasi-optimal $L^\infty(\Omega)$-norm error estimate is also obtained. Further, we analyze two fully discrete schemes using convolution quadrature in time based on the backward Euler and the second-order backward difference methods, and derive error estimates for smooth and nonsmooth data. Finally, we present several numerical examples to confirm our theoretical results.

1. Introduction

Let Ω be a bounded convex polygonal domain in $\mathbb{R}^2$ with a boundary $\partial\Omega$ and $T > 0$ be a fixed value. Let v be a given function defined on Ω. We consider the initial boundary-value problem for the time-fractional evolution equation

$$u_t(x,t) = -\gamma\, \partial_t^{1-\alpha_1} u(x,t) + \partial_t^{1-\alpha_2} \Delta u(x,t) \quad \text{in } \Omega \times (0,T], \tag{1.1a}$$

with the homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{on } \partial\Omega \times (0,T], \tag{1.1b}$$

and the initial condition

$$u(x,0) = v(x) \quad \text{in } \Omega, \tag{1.1c}$$

where $0 < \alpha_1 < 1$, $0 < \alpha_2 < 1$ and $\gamma > 0$ are constants, u_t is the partial derivative of u with respect to time, and $\partial_t^{1-\alpha} := {}^R\mathrm{D}^{1-\alpha}$ is the Riemann–Liouville fractional derivative in time defined for $0 < \alpha < 1$ by:

$$\partial_t^{1-\alpha} \varphi(t) := \frac{d}{dt} \int_0^t \omega_\alpha(t-s)\varphi(s)\, ds \quad \text{with} \quad \omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \tag{1.2}$$

where $\Gamma(\cdot)$ is the Gamma function.

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[1] This research is supported by The Research Council of Oman under grant ORG/CBS/15/001.

https://doi.org/10.1016/j.apnum.2018.05.012
0168-9274/

Please cite this article in press as: M. Al-Maskari, S. Karaa, The lumped mass FEM for a time-fractional cable equation, Appl. Numer. Math. (2018), https://doi.org/10.1016/j.apnum.2018.05.012

In recent years time-fractional evolution equations received considerable attention from both practical and theoretical points of view due to their various applications. Fractional models have shown great efficiency in describing anomalous diffusion observed in many physical situations, see [9], [10] and [25]. One particular motivating example is the fractional cable equation which is a generalization of the classical cable equation obtained by taking into account the anomalous diffusion in the movement of the ions in neuronal system. In [17], Langlands et al. derived such an equation as a macroscopic model for electrodiffusion of ions in nerve cells, when molecular diffusion is anomalous subdiffusion due to binding, crowding or trapping. They also investigated the equation for modeling the anomalous diffusion in spiny neuronal dendrites [11].

Recently, several numerical studies have been devoted to the approximation of the fractional cable equation. In [18], Lin et al. proposed a scheme based on a finite difference approach in time and a spectral method in space. A detailed error analysis is presented and some numerical examples are provided to verify the theoretical results. Bhrawy and Zaky [2] considered an accurate spectral collocation algorithm for solving nonlinear fractional cable equations in one- and two-dimensional cases. Zhuang et al. [29] discussed a Galerkin finite element approximation for one-dimensional fractional cable equation with first-order accuracy in time. Hu and Zhang [12] constructed implicit compact finite difference schemes for the approximation of the fractional cable equation. In [26], Quintana-Murillo and Yuste proposed an explicit numerical method for the fractional cable equation using the Grunwald–Letnikov formula for the temporal Riemann–Liouville derivative. In the recent paper [28], a discontinuous Galerkin finite element method was investigated. In all these studies, error estimates are obtained assuming high regularity on the exact solution, which is not practically the case. Most recently, Zhu et al. [30] applied the standard Galerkin FEM to approximate the solution of (1.1), and derived optimal error estimates with respect to the solution smoothness, expressed through the initial data v. They also established optimal error bounds for the inhomogeneous problem.

When $\gamma = 0$, the model (1.1a) reduces to the time-fractional subdiffusion equation

$$u_t(x,t) - \partial_t^{1-\alpha_2} \Delta u(x,t) = 0, \tag{1.3}$$

for which numerical schemes with optimal (with respect to the regularity of problem data) error estimates are available. In [23], McLean and Thomée established the first optimal $L^2(\Omega)$-error estimates for the Galerkin FE solution of (1.3) with respect to the regularity of initial data. More precisely, for $t \in (0,T]$, convergence rates of order $t^{\alpha(\delta-2)/2}h^2$ (h denoting the maximum diameter of the spatial mesh elements) were proved assuming that the initial data $u_0 \in \dot{H}^\delta(\Omega)$ for $\delta = 0, 2$ (see, Section 2 for the definition of these spaces). The proof was based on some refined estimates of the Laplace transform in time for the error. The estimate extends results obtained for the standard parabolic problem, i.e., $\alpha = 1$, which has been thoroughly studied, see [27]. In [24], based on a similar approach, the authors derived $O(t^{-\alpha(2-\delta)/2}h^2|\ln h|^2)$ convergence rates in the stronger $L^\infty(\Omega)$-norm. Recently, a delicate energy analysis has been developed in [15] to obtain similar estimates. In recent papers [13,14], Jin et al. considered a subdiffusion model involving a Caputo fractional derivative in time. They established *a priori* error estimates for a semidiscrete FE scheme for the homogeneous problem in [13]. Fully discrete schemes based on convolution quadrature in time were derived and analyzed in [14] for problems with smooth and nonsmooth data.

The goal of this paper is to develop an error analysis with optimal (with respect to the regularity of problem data) estimates for the lumped mass Galerkin FEM for problem (1.1), using piecewise linear functions. An important feature of the lumped mass method is that when representing the discrete solution in the nodal basis functions, the resulting mass matrix is diagonal, which enhances the computation procedure. Our analysis is based on a comparison of the Galerkin FE solution with the lumped mass FE solution and exploiting Laplace transforms tool with semigroup type properties of the FE solution operator. The idea of comparing the two solutions has been used in [3] and [4] for the approximation of the standard parabolic problem by the lumped mass FE method and the FVE method, respectively, leading to an improvement of earlier results [5]. Recently in [16], a unified finite volume element error analysis has been presented for a large class of time-fractional evolution problems.

Our second objective is to analyze two fully discrete schemes for the semidiscrete FE problem based on convolution quadrature in time generated by the backward Euler and the second-order backward difference methods. Error estimates with respect to the data regularity are provided in Theorems 5.1 and 5.2.

The rest of the paper is organized as follows. In the next section, we present the solution theory of the mathematical model (1.1) and derive properties of the solution operator, which will play an important role in our subsequent error analysis. In Section 3, we derive smooth and nonsmooth data error estimates for the standard Galerkin method, which will be used in the sequel. In Section 4, we describe the semidiscrete lumped mass scheme, recall some properties, and establish error estimates for smooth and nonsmooth initial data $v \in \dot{H}^q$, $q = 0, 1, 2$ in Subsections 4.1 and 4.2. For $q = 0$, i.e., $v \in L^2(\Omega)$, we show an optimal error bound under an additional assumption on the mesh. Superconvergence result is proved in Subsection 4.3 and as a consequence, a quasi-optimal error estimate is established in the $L^\infty(\Omega)$-norm. In Section 5, two fully discrete schemes based on convolution quadrature approximation of the fractional derivative are presented and error estimates are established. Finally, in Section 6, we conduct numerical experiments to validate our theoretical results.

Throughout the paper, c denotes a generic positive constant that may depend on α_1, α_2 and γ, but are independent of the spatial mesh element size h.