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Two-grid methods for expanded mixed finite element approximations of semi-linear parabolic integro-differential equations

Tianliang Hou [a], Luoping Chen [b,*], Yin Yang [c]

[a] School of Mathematics and Statistics, Beihua University, Jilin 132013, Jilin, China
[b] School of Mathematics, Southwest Jiaotong University, Chengdu 611756, Sichuan, China
[c] School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, China

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ABSTRACT

In this paper, we investigate a two grid discretization scheme for semilinear parabolic integro-differential equations by expanded mixed finite element methods. The lowest order Raviart–Thomas mixed finite element method and backward Euler method are used for spatial and temporal discretization respectively. Firstly, expanded mixed Ritz–Volterra projection is defined and the related a priori error estimates are proved. Secondly, a superconvergence property of the pressure variable for the fully discretized scheme is obtained. Thirdly, a two-grid scheme is presented to deal with the nonlinear part of the equation and a rigorous convergence analysis is given. It is shown that when the two mesh sizes satisfy $h = H^2$, the two grid method achieves the same convergence property as the expanded mixed finite element method. Finally, a numerical experiment is implemented to verify theoretical results of the two grid method.

1. Introduction

In this paper, we consider the following semi-linear parabolic integro-differential equations

$$y_t - \operatorname{div}(A(t)\nabla y) + \int_0^t \operatorname{div}(B(t,s)\nabla y(s))ds = f(y), \quad x \in \Omega,\ t \in J, \tag{1.1}$$

$$y(x,t) = 0, \quad x \in \partial\Omega,\ t \in J, \tag{1.2}$$

$$y(x,0) = y_0(x), \quad x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbf{R}^2$ is a convex polygonal domain with the boundary $\partial\Omega$, $J = (0,T]$, $f(y) = f(y,x,t)$ is a given real-valued function on Ω. We assume that the coefficient matrix $A(t) = A(x,t) = (a_{ij}(x,t))_{2\times 2} \in W^{1,\infty}(\bar{\Omega}; \mathbf{R}^{2\times 2})$ is a symmetric and positive definite 2×2-matrix, and there exist constants $a_1, a_2 > 0$ satisfying for any vector $\mathbf{X} \in \mathbf{R}^2$, $a_1\|\mathbf{X}\|^2_{\mathbf{R}^2} \le \mathbf{X}^t A\mathbf{X} \le a_2\|\mathbf{X}\|^2_{\mathbf{R}^2}$. Moreover, $B(t,s) = B(x,t,s)$ is also a 2×2 matrix. We also assume that

* Corresponding author.
E-mail addresses: htlchb@163.com (T. Hou), cherrychen@home.swjtu.edu.cn (L. Chen), yangyinxtu@xtu.edu.cn (Y. Yang).

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$$|f'(y)| + |f''(y)| \le M, \quad y \in \mathbf{R}.$$

Integro-differential equations can arise from many physical processes in which it is deficiency (the local characteristic) of the usual diffusion equations (see, for example [10]). At present, many numerical methods have been developed for solving these problems. The mathematical difficulty associated with the analysis of numerical approximations to the solution of integro-differential equations lies on the integral term. For linear or nonlinear integro-differential equations, the standard finite element methods have received considerable attention and are well studied (see, for example [10,11,18,23,27]). For mixed finite element methods, please see the References [15,16,22,26]. In [15], a sharper L^2-error estimate is obtained for the non-Fickian flow of fluid in porous media by means of a mixed Ritz–Volterra projection instead of the mixed elliptic projection. Subsequently, Ewing et al. [16] have derived maximum norm estimates and superconvergence results for mixed semidiscrete approximation to partial integro-differential equations using mixed Ritz–Volterra projection and a tool of approximate Green's function. In [22], Jiang derived the $L^\infty(L^2)$ and $L^\infty(L^\infty)$ error estimates for mixed finite element methods for integro-differential equations of parabolic type. In [26], Sinha et al. analyzed the semidiscrete mixed finite element methods for parabolic integro-differential equations which arise in the modeling of nonlocal reactive flows in porous media and obtained a priori L^2 error estimates for pressure and velocity are obtained with both smooth and nonsmooth initial data.

The two-grid method which was first proposed by Xu [29,30] as a discretization method for nonsymmetric, indefinite and nonlinear partial differential equations. The main idea is to use a coarse-grid space to produce a rough approximation of the solution of nonlinear problems, and then use it as the initial guess for one Newton-like iteration on the fine grid. In [29,30], Xu mainly considered two-grid finite element method for semi-linear elliptic and parabolic problems. After his work, two-grid method combined with other numerical methods were further investigated by many authors (see, e.g., [3,4,6–8,12,28,31]). Wu and Allen [28] established and analyzed a two-step algorithm by using the two-grid idea for the semi-linear reaction–diffusion equations with the expanded mixed finite element method. Based on this work, in order to improve accuracy of the algorithm, Chen et al. [8,7] proposed some multi-step two-grid mixed finite element algorithms for semilinear parabolic equations. Chen et al. [6] discussed a two-grid method for mixed finite element methods of fully nonlinear reaction–diffusion equations. Dawson and Wheeler [12,13] applied this method combined with finite difference method and mixed finite element method to nonlinear parabolic equations. Bi and Ginting [3] studied two-grid finite volume element method for linear and nonlinear elliptic problems. They also investigated two-grid discontinuous Galerkin method for quasi-linear elliptic problems in [4]. Chen, Yang and Bi [5] considered two-grid finite volume element method for semi-linear parabolic equations. Xu and Zhou [31] presented a two-grid discretization scheme for eigenvalue problems. In [21], Jin, Shu and Xu proposed a two-grid finite element method for solving coupled partial differential equations, e.g., the Schrödinger-type equation. There also exist other efficient methods such as multilevel algorithm for nonlinear elliptic equations and Ginzburg–Landau model, (see, e.g., [19,20]). However, as far as we know there is no convergence analysis of two-grid expanded mixed finite element method for nonlinear parabolic integro-differential equations in the literature. The nonlinear term makes the theoretical analysis and numerical experiment more difficult. In this paper, we will use the two-grid method combined with the lowest order expanded Raviart–Thomas mixed finite element method to solve problem (1.1)–(1.3). We first solve a nonlinear problem on the coarse-grid space, then we use the known coarse grid solution and a Taylor expansion to extrapolate the solution on the fine grid. On the fine grid we only need to solve a linear system. As shown in [6,29], the coarse mesh can be quite coarse and still maintain a good accuracy approximation.

The plan of this paper is as follows. In section 2, we will make some necessary preparations, define the expanded mixed Ritz–Volterra projection, and construct the fully discretized expanded mixed finite element approximation of the problem (1.1)–(1.3). In section 3, we analyze the approximation properties of the expanded mixed Ritz–Volterra projection and a superconvergence property of the pressure variable for the fully discretized scheme. In Section 4, we present the two-grid algorithm and prove its error estimates. In Section 5, a numerical experiment is given to verify the theoretical results completely. In the last section, we briefly summarize the results obtained and some possible future extensions.

2. Fully discretized mixed finite element scheme

In this section, we shall present the fully discretized expanded mixed finite element approximation scheme of the problem (1.1)–(1.3).

We adopt the standard notation $W^{m,p}(\Omega)$ for Sobolev spaces (see[1]) on Ω with a norm $\|\cdot\|_{m,p}$ given by $\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{L^p(\Omega)}^p$, a semi-norm $|\cdot|_{m,p}$ given by $|v|_{m,p}^p = \sum_{|\alpha|=m} \|D^\alpha v\|_{L^p(\Omega)}^p$. We set $W_0^{m,p}(\Omega) = \{v \in W^{m,p}(\Omega) : v|_{\partial\Omega} = 0\}$. For $p = 2$, we denote $H^m(\Omega) = W^{m,2}(\Omega)$, $H_0^m(\Omega) = W_0^{m,2}(\Omega)$, and $\|\cdot\|_m = \|\cdot\|_{m,2}$, $\|\cdot\| = \|\cdot\|_{0,2}$.

We denote by $L^s(J; W^{m,p}(\Omega))$ the Banach space of all L^s integrable functions from J into $W^{m,p}(\Omega)$ with norm $\|v\|_{L^s(J;W^{m,p}(\Omega))} = \left(\int_0^T ||v||_{W^{m,p}(\Omega)}^s dt\right)^{\frac{1}{s}}$ for $s \in [1, \infty)$, and the standard modification for $s = \infty$. For simplicity of presentation, we denote $\|v\|_{L^s(J;W^{m,p}(\Omega))}$ by $\|v\|_{L^s(W^{m,p})}$. Similarly, one can define the spaces $H^1(J; W^{m,p}(\Omega))$ and $C^k(J; W^{m,p}(\Omega))$. In addition C denotes a general positive constant independent of h and Δt, where h is the spatial mesh-size and Δt is time step.