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Convergence estimates for multigrid algorithms with SSC smoothers and applications to overlapping domain decomposition

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ABSTRACT

In this paper we study convergence estimates for a multigrid algorithm with smoothers of successive subspace correction (SSC) type, applied to symmetric elliptic PDEs under no regularity assumptions on the solution of the problem. The proposed analysis provides three main contributions to the existing theory. The first novel contribution of this study is a convergence bound that depends on the number of multigrid smoothing iterations. This result is obtained under no regularity assumptions on the solution of the problem. A similar result has been shown in the literature for the cases of full regularity and partial regularity assumptions. Second, our theory applies to local refinement applications with arbitrary level hanging nodes. More specifically, for the smoothing algorithm we provide subspace decompositions that are suitable for applications where the multigrid spaces are defined on finite element grids with arbitrary level hanging nodes. Third, global smoothing is employed on the entire multigrid space with hanging nodes. When hanging nodes are present, existing multigrid strategies advise to carry out the smoothing procedure only on a subspace of the multigrid space that does not contain hanging nodes. However, with such an approach, if the number of smoothing iterations is increased, convergence can improve only up to a saturation value. Global smoothing guarantees an arbitrary improvement in the convergence when the number of smoothing iterations is increased. Numerical results are also included to support our theoretical findings.

1. Introduction

Multigrid algorithms have been introduced in the literature since the 1960's with pioneering works such as [2,10,22,31]. A wide literature of both theoretical and computational works has been developed ever since, see for example [3,9,13,29, 32–34,38–40,42] and the references therein. Convergence proofs of multigrid algorithms usually rely on two properties referred to as the smoothing and the approximation property [4,5,24]. The former is related to the definition of the smoothing operator involved in the algorithm, while the latter is usually proved assuming full elliptic regularity for the solution of the partial differential equation considered. A breakthrough in the convergence analysis took place with [8], where the elliptic regularity assumption has been dropped. The error bound obtained in [8] is not optimal, in the sense that it becomes worse

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as the number of multigrid levels increases; moreover, no dependence of the bound on the number of smoothing iterations is shown. A subsequent analysis under no regularity assumptions was carried out in [6], where convergence estimates for the case of a multigrid algorithm were obtained with non-symmetric subspace correction smoothers. The setup in [6] is similar to the one described in this paper, but for local refinement applications, a local smoothing procedure was adopted, where smoothing was carried out only on a subspace of the multigrid space. Moreover, the error bound obtained in [6] for applications to both uniform refinement and local refinement with hanging nodes showed no dependence on the number of smoothing steps. To the best of our knowledge, such a paper is the only work in the literature that presents a multigrid algorithm with subspace correction smoother. A unifying scheme encompassing successive subspace correction algorithms is given by Xu in [41]. Further work has been done by Bramble and Pasciak in [7], where optimal convergence was proved, assuming *partial* regularity of the solution. However, no dependence on the number of smoothing iterations was reported yet. An improvement in addressing this matter was made in [11]. The author showed that, again under partial regularity assumptions, the multigrid error bound is optimal and can be improved increasing the number of smoothing iterations. The results obtained in [11] for a Richardson relaxation scheme were extended in [12] to the Jacobi and Gauss–Seidel smoothers. More recently, further work on multigrid methods that rely on minimal regularity assumptions has been done in [16], where graded meshes obtained by a variant of the newest vertex bisection method are considered. The present work contributes to the existing multigrid literature with a multigrid error bound that depends on the number of smoothing iterations when no regularity assumptions are made on the solution. This is a new results for the framework of no regularity assumptions. The second new contribution of the present work is that our analysis applies to local refinement applications with arbitrary hanging nodes configurations. Uniform refinement results are also obtained, that upgrade the ones in [6]. For the local refinement applications, we derive ad-hoc decompositions of the finite element spaces and set suitable choices of approximate subdomain solvers. Two local refinement cases are considered. In the first, we construct a decomposition that exploits a multilevel approach and has the advantage of being easy to implement and suitable for standard finite element codes. However, it does not allow freedom in the choice of the subdomains on which the subspaces are built. The second local refinement case relies on a domain decomposition approach and requires additional work to ensure continuity of the finite element solution; namely it requires a non-standard finite element implementation. Such a decomposition enables a choice of the subdomains that does not depend on the multigrid level. A third contribution to the existing multigrid theory is the use of global smoothing at each multigrid level. Whenever hanging nodes are present, many existing procedures choose local smoothing on a subspace of the multigrid space, in particular we report [6,8,28]. A global smoothing procedure allows an arbitrary improvement of the convergence when the number of smoothing iterations is increased. Such an arbitrary improvement cannot be achieved with local smoothing.

The outline of the paper is as follows. In Section 2 we briefly recall the multigrid V-cycle algorithm and then we present our novel convergence analysis based on three assumptions on the smoothing error operator. This analysis needs no regularity of the solution. Section 3 illustrates the algorithm used for the smoothing iteration and shows how it can be related to the multigrid convergence theory. Uniform and local refinement applications of the analysis described in the previous sections are presented in Section 4, where convergence bounds are obtained for the specific cases. In Section 5, we present some numerical experiments to support our theoretical findings. Finally, we draw our conclusions.

2. A multigrid convergence analysis under no regularity assumptions

This section describes our new convergence analysis based on no regularity assumptions. We start by briefly recalling the multigrid algorithm.

Throughout the paper, the total number of levels is denoted as J. For $k = 0, \dots, J$, let V_k be a finite-dimensional vector space such that

$$V_0 \subset V_1 \subset \cdots \subset V_J\,, \tag{1}$$

and let $(\cdot,\cdot)$ and $a(\cdot,\cdot)$ be two symmetric positive definite (SPD) bilinear forms on V_k. Hence, both bilinear forms are inner products on V_k. Let $||\cdot|| = \sqrt{(\cdot,\cdot)}$ and $||\cdot||_E = \sqrt{a(\cdot,\cdot)}$ be the corresponding induced norms. Associated with these inner products, let us also define the operators $Q_k : V_J \to V_k$ and $P_k : V_J \to V_k$ as the orthogonal projections with respect to $(\cdot,\cdot)$ and $a(\cdot,\cdot)$, respectively; namely, for all $v \in V_J$ and all $w \in V_k$

$$(Q_k v, w) = (v, w)\,, \quad a(P_k v, w) = a(v, w)\,. \tag{2}$$

Note that from the definition of P_k it follows that

$$a((I - P_k)v, w) = 0 \quad \text{for all} \quad w \in V_k\,. \tag{3}$$

The multigrid algorithm seeks solutions of the following problem: given $f \in V_J$, find $u \in V_J$ such that

$$a(u, v) = (f, v) \quad \text{for all} \quad v \in V_J\,. \tag{4}$$

For $k = 0, \dots, J$, define the operator $A_k : V_k \to V_k$ as