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A high-order source removal finite element method for a class of elliptic interface problems

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ABSTRACT

A high-order finite element method based on unfitted meshes for solving a class of elliptic interface problems whose solution and its normal derivative have finite jumps across an interface is proposed in this paper. The idea of the method is based on the source removal technique first introduced in the immersed interface method (IIM). The strategy is to use the level set representation of the interface and extend the jump conditions that are defined along the interface to a neighborhood of the interface. In our numerical method, the jump conditions only need to be extended to the Lagrange points of elements intersecting with the interface. Optimal error estimates of the method in the broken H^1 and L^2 norms are rigorously proven. Numerical examples presented in this paper also confirm our theoretical analysis.

1. Introduction

In this paper, we consider high-order finite element approximations to the following interface problem:

$$-\Delta u^{\pm}(x) = f^{\pm}(x) \quad \text{in } \Omega^{\pm}, \tag{1.1a}$$

$$u(x) = 0 \quad \text{on } \partial\Omega, \tag{1.1b}$$

$$[u](X) = w_0(X) \quad \text{on } \Gamma, \tag{1.1c}$$

$$\left[\frac{\partial u}{\partial \mathbf{n}}\right](X) = w_1(X) \quad \text{on } \Gamma, \tag{1.1d}$$

where $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ is a convex polygonal or polyhedral domain and $\Omega^{\pm}$ are two disjoint subdomains of Ω separated by a smooth and closed interface Γ. Without loss of generality, we assume Ω^- lies strictly inside Ω, that is, $\Gamma = \partial\Omega^-$. The jump conditions of the solution across the interface Γ are defined as

$$[u] = u^+ - u^-, \qquad \left[\frac{\partial u}{\partial \mathbf{n}}\right] = \nabla u^+ \cdot \mathbf{n} - \nabla u^- \cdot \mathbf{n},$$

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where $u^{\pm} = u|_{\Omega^{\pm}}$ and $\mathbf{n}$ is the unit vector normal to Γ pointing towards Ω^{+}. In general, we use x to represent a point in the domain, and X a point on the interface. The interface problem (1.1) can be rewritten as a single equation in the whole domain but with singular sources. For example, if $w_0 = 0$, then the interface problem (1.1) is equivalent to

$$-\Delta u(x) = f(x) - \int_{\Gamma} w_1(X(s))\delta(x - X(s))ds \qquad \forall x \in \Omega, \tag{1.2}$$

where δ is the Dirac delta function.

The interface problem (1.1) arises in many applications. For example, we encounter this kind of interface problem when we apply the projection method for solving Navier–Stokes or Stokes equations with a surface force (e.g., surface tension) that acts on an immersed interface. The jump conditions of the pressure and the velocity can be expressed in terms of the surface force [20,36]. In the electrostatic field computations, we also encounter this kind of interface problems in which the interface data w_1 refers to the surface charge density [15]. In addition, the discretization of the interface problem (1.1) is a key step in the augmented finite difference/element method [21,18], the kernel-free boundary integral method [37,38] and the boundary integral method [27,28] for solving irregular domain problems and interface problems with discontinuous coefficients. In the augmented method, the interface data w_0 or w_1 is chosen as an augmented variable. The augmented variable (interface data) should be chosen so that the solution satisfies original interface or boundary conditions and is solved by using the GMRES method. While in the boundary integral method, the interface data is obtained by solving corresponding boundary integral equations.

For the partial differential equations (PDEs) involving immersed interfaces, existing numerical methods generally can be classified as two categories: the fitted mesh methods in which the mesh is aligned with the interface; and unfitted mesh methods in which the mesh is generated independently of the interface and allows the interface to cut through the mesh. In implementation, it may be difficult and time consuming to generate a fitted mesh for the problem with a complicated interface. Such a difficulty may become even severer for moving interface problems because a new fitted mesh has to be generated at each time step and an interpolation scheme is required to transfer the numerical solutions between different meshes. From this point of view, it would be preferable to use an unfitted mesh in which the interface can be arbitrarily located with respect to the fixed background mesh. The natural unfitted mesh is a Cartesian mesh. The goal of this work is to develop a high-order finite element method based on the unfitted meshes.

There are many numerical methods using unfitted meshes for solving the interface problem (1.1). One example is Peskin's immersed boundary (IB) method [32] in which the interface jump condition is treated as a singular source and then a discrete delta function is used. The IB method is simple and robust but it is only first-order accurate. Recently, Li [22] gave a rigorous analysis of the IB method for the above elliptic interface problem. To improve the accuracy, LeVeque and Li [19] developed a second-order Cartesian grid method called the immersed interface Method (IIM) in which the interface jump conditions were incorporated into the finite difference scheme. It has been proved in [1] that not only the solution but also the gradient of the IIM are second-order accurate. In [16], a fourth order Cartesian grid method was proposed for elliptic PDEs on irregular domains. In the finite element method field, recently Guzmán et al. [7] developed an edge-based correction finite element method for the interface problem (1.1) in two dimensions and proved that the method is second-order accurate. Using the abstract error analysis established in [7], the authors also proved the optimal convergence of the immersed finite element method (IFEM) proposed in [8] for solving the interface problem (1.1) with $w_0 = 0$. However, the idea of the edge-based correction can not be easily extended to high-order methods or three-dimensional interface problems. More recently, Guzmán et al. [6] developed a high-order finite element method for the interface problem (1.1) in two dimensions. In this method, high order interface jump conditions are derived and enforced on several points on the interface exactly to construct local finite element spaces on interface elements. The authors also proved optimal error estimates of the methods on general quasi-uniform and shape regular meshes in maximum norms. However, the extension of the method to three-dimensional interface problems seems complicated and hard to analyze, for example, the intersection points of the interface and mesh may not lie on a plane for three-dimensional cases. Note that for interface problem with discontinuous coefficients there are many new developed methods in the literature such as partially penalized immersed finite element methods [26], Petrov–Galerkin finite element methods based nonsymmetric weak formulations [11,13,14], weak Galerkin methods [29,30], etc.

In this work, we develop and analyze a high-order finite method for solving the interface problem (1.1) in two and three dimensions. Our method has the feature that the stiffness matrix is the same as that obtained by standard high-order piecewise continuous finite element methods and only the right-hand side needs to be modified on interface elements. The modification is based on a correction function constructed by using the source removal technique. The technique was first introduced by Li et al. [25] to deal with the jumps of solutions across interfaces. Since then, the source removal technique has been applied to the finite element framework which is only restricted to linear finite elements (see [17,3,5]). In the source removal technique, a function that satisfies the same interface conditions across the interface is constructed using the level set representation of the interface and extensions of the interface data. With this function, the discontinuities in the solution and its normal derivative are removed. For high-order methods, the function should be constructed to satisfy higher order interface conditions which can be derived by using both the original interface conditions and the original PDE. In our high-order finite element method, the correction function in the modification of the right-hand side is defined by using such a constructed function. We emphasize that we only need the values of the constructed function at Lagrange