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Multistep collocation approximations to solutions of first-kind Volterra integral equations ☆

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ABSTRACT

The multistep collocation method is applied to Volterra integral equations of the first kind. The existence and uniqueness of the multistep collocation solution are proved. Then the convergence condition of the multistep collocation method is analyzed and the corresponding convergence order is described. In particular, for $c_m = 1$, the convergence conditions, which can be easily implemented, are given for two-step and three-step collocation methods. Numerical experiments illustrate the theoretical analysis.

1. Introduction

Recently, for Volterra integral equations (VIEs) of the second kind, a general class of multistep collocation methods is presented in [4]. Later on, in [5], two-step diagonally-implicit collocation methods are investigated; in [7,8], super implicit multistep collocation methods and multistep Hermite collocation methods are studied respectively; in [13], collocation methods by continuous piecewise polynomial collocation approximations are analyzed, which correspond to the two-step collocation method investigated in [4]. In addition, for Volterra integro-differential equations, multistep collocation methods are analyzed in [3], and superimplicit multistep collocation methods are studied in [6].

In this paper, we consider the following VIE of the first kind

$$\int_0^t K(t,s)y(s)ds = f(t), \quad t \in I := [0,T]. \tag{1.1}$$

Here f and K are supposed to be sufficiently smooth functions satisfying $f(0) = 0$ and $|K(t,t)| \geqslant k_0 > 0$ for all $t \in I$. By the proof of [1, Theorem 2.1.8], we know that (1.1) can be reformulated as a VIE of the second kind, which means that (1.1) can be solved numerically after the reformulation. But we prefer direct solving of (1.1), since now the differentiation and approximation of derivatives of $K(t,s)$ and $f(t)$ are not needed. There are many results on the convergence analysis of

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one-step piecewise polynomial collocation methods for the first-kind VIE (1.1); see the monographs [1,2] and the references cited therein for discontinuous collocation approximations. Furthermore, in [11,10], the multistep methods for the first-kind VIE (1.1) are analyzed and sufficient conditions for multistep methods to be convergent are derived. Especially, in [12], the continuous collocation approximations to the first-kind VIE (1.1) is studied, which is also exactly the two-step collocation method investigated in this paper. However, as far as we know, up to now, there are no papers focusing on multistep (more than two-step) collocation methods for VIE of the first kind, and it is the aim of this paper to present a complete analysis for it.

2. Construction of the multistep collocation method

In order to seek the multistep collocation method for first-kind VIE (1.1), let us discretize the interval I by introducing a uniform mesh $I_h := \{t_n := nh, n = 0, 1, \ldots, N (t_N := T)\}$ with mesh diameter $h := \frac{T}{N}$ and $N \geq 2$ being an integer. Define the subintervals $\sigma_0 := [t_0, t_1]$ and $\sigma_n := [t_n, t_{n+1}]$, $n = 1, \ldots, N-1$. Let the collocation parameters be $0 < c_1 < \cdots < c_m \leq 1$ and the collocation points be $t_{n,j} := t_n + c_j h$, $j = 1, \ldots, m$; $n = 0, \ldots, N-1$.

The $r+1$-step collocation method is obtained by introducing in the collocation polynomial the dependence on r previous time steps; namely we seek an approximation u_h to the solution y of (1.1), which is represented by the interpolation formula

$$u_h(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) Y_{n,j}, \quad s \in [0,1],\ n \geq r-1, \tag{2.1}$$

where $\varphi_k(s)$ and $\psi_j(s)$ are polynomials of degree $m + r - 1$, and $Y_{n,j} := u_h(t_{n,j})$, $y_{n-k} := u_h(t_{n-k})$. In σ_n $(0 \leq n < r-1)$, the starting values $y_1, y_2, \ldots, y_{r-1}$ can be obtained based on a classical one step method, and y_0 is chosen as the exact value $y(0) = y(t_0) = \frac{f'(0)}{K(0,0)}$ by differentiating with respect to t in (1.1).

The interpolation conditions at t_{n-k}, $k = 0, \ldots, r-1$, that is $y_{n-k} = u_h(t_{n-k})$, together with the condition $Y_{n,j} = u_h(t_{n,j})$, lead to the following linear system:

$$\varphi_l(-k) = \delta_{lk},\ \varphi_l(c_j) = 0, \quad l, k = 0, \ldots, r-1;$$

$$\psi_i(-k) = 0,\ \psi_i(c_j) = \delta_{ij}, \quad i, j = 1, \ldots, m.$$

By [4], we know that

$$\begin{cases} \varphi_k(s) = \displaystyle\prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \frac{s+i}{-k+i}, \\ \psi_j(s) = \displaystyle\prod_{i=0}^{r-1} \frac{s+i}{c_j+i} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{s - c_i}{c_j - c_i}. \end{cases} \tag{2.2}$$

Moreover, the approximation u_h satisfies the following collocation equation

$$f(t_{n,j}) = \int_0^{t_{n,j}} K(t_{n,j}, s) u_h(s) ds, \quad j = 1, 2, \ldots, m. \tag{2.3}$$

Inserting (2.1) into (2.3), we obtain, for each $n = r-1, \ldots, N-1$, a linear system for the unknown $Y_{n,j}$ with $j = 1, \ldots, m$:

$$\begin{aligned} &h \sum_{j=1}^{m} \int_0^{c_i} K(t_{n,i}, t_n + sh) \psi_j(s) ds Y_{n,j} \\ &= f(t_{n,i}) - h \sum_{l=0}^{r-2} \int_0^1 K(t_{n,i}, t_l + sh) u_h(t_l + sh) ds \\ &\quad - h \sum_{l=r-1}^{n-1} \sum_{k=0}^{r-1} \int_0^1 K(t_{n,i}, t_l + sh) \varphi_k(s) ds y_{l-k} \end{aligned}$$