Applied Numerical Mathematics 129 (2018) 39–57

Contents lists available at ScienceDirect

Applied Numerical Mathematics

www.elsevier.com/locate/apnum

Error analysis of the spectral element method with Gauss–Lobatto–Legendre points for the acoustic wave equation in heterogeneous media

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ARTICLE INFO

Article history:
Received 1 September 2017
Received in revised form 19 February 2018
Accepted 20 February 2018
Available online xxxx

Keywords:
Spectral element method
Acoustic wave equation
A priori error analysis
Heterogeneous media

ABSTRACT

We present the error analysis of a high-order method for the two-dimensional acoustic wave equation in the particular case of constant compressibility and variable density. The domain discretization is based on the spectral element method with Gauss–Lobatto–Legendre (GLL) collocation points, whereas the time discretization is based on the explicit leapfrog scheme. As usual, GLL points are also employed in the numerical quadrature, so that the mass matrix is diagonal and the resulting algebraic scheme is explicit in time. The analysis provides an a priori estimate which depends on the time step, the element length, and the polynomial degree, generalizing several known results for the wave equation in homogeneous media. Numerical examples illustrate the validity of the estimate under certain regularity assumptions and provide expected error estimates when the medium is discontinuous.

1. Introduction

The spectral element method, a high-order method that combines the high accuracy of spectral methods and the geometrical flexibility of finite elements, was originally proposed for fluid dynamics [24], and has reached great success in the numerical simulation of seismic waves [19,21,25,29]. In this field, the spectral element method with Gauss–Lobatto–Legendre (GLL-SEM) collocation points is most often used because it naturally leads to explicit time-marching schemes.

Maday and Rønquist [22] carried out a p analysis of GLL-SEM for the diffusion equation with variable coefficients and non-affine elements. The analysis for Stokes and Navier–Stokes equations can be found in [4]. The error analysis for the wave equation in homogeneous media was presented by Zampieri and Pavarino [32], Rong and Xu [27], and Durufle et al. [11]. The first two works carry out the study entirely in time domain, following the steps from [26], while the analysis in [11] is developed in the Laplace transform domain.

In this work we extend error estimates reported in [11,27,32] to the following scalar wave equation:

$$\ddot{u}(\boldsymbol{x},t) - \operatorname{div}(c^2(\boldsymbol{x})\,\nabla u(\boldsymbol{x},t)) = f(\boldsymbol{x},t). \tag{1}$$

Error analyses of continuous and discontinuous Galerkin methods for equation (1) have been proposed in the literature [9,14]. In the context of acoustic waves, $u(\boldsymbol{x},t)$ models the acoustic pressure in a medium with heterogeneous density $\rho(\boldsymbol{x})$

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https://doi.org/10.1016/j.apnum.2018.02.007

and constant bulk modulus K_0, and $c(\boldsymbol{x}) = \sqrt{K_0/\rho(\boldsymbol{x})}$ represents the speed of propagation. In general, heterogeneity of physical parameters is a crucial assumption on wave propagation models. Discontinuous coefficients are frequently employed, for instance in the description of geological structures, but smooth parameters are also relevant in wave propagation, especially with the advent of functionally graded materials [8,16,23].

Analogously to the study presented in [27,32], the analysis is performed in time domain, but since our focus is in the leapfrog time integration, our analysis is structured as in the study by Grote and Schötzau of an interior penalty discontinuous Galerkin method [14]. Their theoretical results cannot be directly applied to GLL-SEM because quadrature error must be taken into account. We undertake this task, gathering relevant results from the literature of *hp* finite element analysis, as well as from the above-mentioned studies of spectral elements. It is worth mentioning that the analysis is limited to homogeneous Dirichlet boundary conditions, which do not represent the typical case of real applications in the framework of wave phenomena [1,20,23].

The remainder of the paper is organized as follows: in the first section we introduce the functional spaces needed in the analysis as well as the wave propagation problem studied in this work and its numerical discretization. In section 3 we present preliminary results for the error analysis. An a priori error bound for the spectral element approximation to the wave equation is provided in Section 4. Section 5 is devoted to numerical examples. Mathematical proofs of auxiliary results are presented in the appendix.

2. Problem setting

Let us initially consider an open, polygonal domain $\Omega \subset \mathbb{R}^2$ with closure $\bar{\Omega}$ and boundary $\Gamma = \partial\Omega$. We consider the Hilbert space $L^2(\Omega)$ equipped with the usual inner product and induced norm

$$(u, v) = \int_{\Omega} u(\boldsymbol{x})v(\boldsymbol{x})\, d\boldsymbol{x}, \quad \|v\|_0^2 = (v, v)^{1/2}. \tag{2}$$

We will denote temporal derivatives as $\dot{u}, \ddot{u}, u^{(3)}, u^{(4)}, \ldots$, whereas spatial derivatives will be denoted as

$$\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots$$

Moreover, given the multi-index $\alpha = (\alpha_1, \alpha_2)$ with $\alpha_1, \alpha_2 \in \mathbb{N}$, we denote

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}, \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2. \tag{3}$$

Given a positive integer m, we will also resort to Sobolev spaces $H^m(\Omega)$ with the standard norms and semi-norms

$$\|v\|_m^2 = \sum_{l=0}^{m} |v|_l^2, \quad |v|_l^2 = \sum_{|\alpha|=l} \|D^\alpha v\|_0^2, \tag{4}$$

as well as Sobolev spaces $W^{m,\infty}(\Omega)$ with norms and semi-norms

$$\|v\|_{m,\infty} = \max_{0\le l\le m} |v|_{l,\infty}, \quad |v|_{l,\infty} = \max_{|\alpha|=l} \operatorname{ess\,sup}\{|D^\alpha v(\boldsymbol{x})|\,;\ \boldsymbol{x} \in \Omega)\}. \tag{5}$$

If $s = m + \theta$, where $m > 0$ is the integer part of s and $0 < \theta < 1$, one can define $H^s(\Omega)$ as follows [13]:

$$H^s(\Omega) = \left\{ v \in H^m(\Omega)\,;\ \|v\|_s^2 = \|v\|_m^2 + \sum_{|\alpha|=m} \int_{\Omega\times\Omega} \frac{|D^\alpha v(\boldsymbol{x}) - D^\alpha v(\boldsymbol{y})|^2}{|\boldsymbol{x} - \boldsymbol{y}|^{2+2\theta}}\, d\boldsymbol{x} d\boldsymbol{y} < \infty \right\}.$$

The following subspace of $H^1(\Omega)$ is crucial in the study of second-order boundary-value problems:

$$H_0^1(\Omega) = \left\{ u \in H^1(\Omega) : u|_\Gamma = 0 \right\},$$

where $u|_\Gamma$ denotes the trace of u on Γ. Following [26, Sec. 7.2], let us introduce spaces of functions $v = v(t)$ mapping elements of the interval $[0, T]$ to a Banach space X with norm $\|\cdot\|_X$. For each integer $m \ge 0$, we denote by $\mathcal{C}^m(0, T; X)$ the space of functions m times continuously differentiable over $[0, T]$ with image into X; this is a Banach space with norm

$$\|v\|_{\mathcal{C}^m(0,T;X)} = \max_{0\le l\le m} \left(\sup_{0\le t\le T} \|v^{(l)}(t)\|_X \right). \tag{6}$$

On the other hand, the space $L^2(0, T; X)$ is defined as the space of functions v strongly measurable over $(0, T)$ with respect to the measure dt. $L^2(0, T; X)$ is a Banach space with norm