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New compact difference scheme for solving the fourth-order time fractional sub-diffusion equation of the distributed order

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ABSTRACT

In this paper, a class of new compact difference schemes is presented for solving the fourth-order time fractional sub-diffusion equation of the distributed order. By using an effective numerical quadrature rule based on boundary value method to discretize the integral term in the distributed-order derivative, the original distributed order differential equation is approximated by a multi-term time fractional sub-diffusion equation, which is then solved by a compact difference scheme. It is shown that the suggested compact difference scheme is stable and convergent in L^∞ norm with the convergence order $\mathcal{O}(\tau^2 + h^4 + (\Delta\gamma)^p)$ when a boundary value method of order p is used, where τ, h and $\Delta\gamma$ are the step sizes in time, space and distributed-order variables, respectively. Numerical results are reported to verify the high order accuracy and efficiency of the suggested scheme. Moreover, in the example, comparisons between some existing methods and the suggested scheme is also provided, showing that our method doesn't compromise in computational time.

1. Introduction

Consider the following initial-boundary value problem

$$\mathbb{D}_t^w u(x,t) + \frac{\partial^4 u}{\partial x^4}(x,t) = f(x,t), \quad 0 < x < L, \ 0 < t \le T, \tag{1.1}$$

$$u(x,0) = 0, \quad 0 \le x \le L, \tag{1.2}$$

$$u(0,t) = u_0(t), \ u(L,t) = u_L(t), \quad 0 \le t \le T, \tag{1.3}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \eta_a(t), \ \frac{\partial^2 u}{\partial x^2}(L,t) = \eta_b(t), \quad 0 \le t \le T, \tag{1.4}$$

where u_0, u_L, η_a, η_b are given functions which satisfy $u_0(0) = u_L(0) = 0$. The fractional derivative $\mathbb{D}_t^w$ of distributed order is defined by

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$$\mathbb{D}_t^w u(x,t) = \int_0^1 w(\gamma)\,{}_0^C D_t^\gamma u(x,t)d\gamma \tag{1.5}$$

with a non-negative continuous function w satisfying

$$\int_0^1 w(\gamma)d\gamma = c_0 > 0, \tag{1.6}$$

and with the Caputo fractional derivative ${}_0^C D_t^\gamma$ defined as

$${}_0^C D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{1}{(t-\nu)^\gamma} \frac{\partial u}{\partial \nu}(x,\nu)d\nu, \quad 0 < \gamma < 1. \tag{1.7}$$

It is worth noting that the zero initial value is only considered here. The numerical method proposed in this paper is also applicable for non-zero initial value problem since the latter can be reduced to a zero initial value one by the use of a proper transformation (cf. [4]).

Equation (1.1) is employed in physics to simulate ultraslow diffusion with a logarithmic growth. Recently much attention, such as existence and uniqueness of fundamental solution, has been paid on such diffusion-like equations with time distributed-order derivative because their application potential in physics, system control and signal processing, see for example [5,12,13,15,17,19] and references therein. Since the analytical solution of the distributed-order equation is not easy to obtain in general, developing efficient numerical method for the solution of distributed-order problem, especially high-order accuracy numerical algorithm, becomes necessary and important. Nevertheless, relatively few works have been done on this topic. And for all we know, most of the existing numerical work deal with the ordinary differential equation of distributed order, see for example [1,7,8]. Here we will briefly review some of these recent advances focusing mainly on the numerical solution of partial differential equation of distributed order. In [23], Ye, Liu and Anh derived and analyzed a compact difference scheme for the time distributed-order diffusion-wave equation on a bounded domain, where the integral term in the distributed-order derivative is discretized by the mid-point quadrature rule and the Caputo fractional derivatives in the resulting multi-term fractional diffusion equation are approximated by the known $L1$ formula (cf. [21]). The similar treatment ways can also be founded in [16] and [14], but where the Caputo fractional derivatives are solved by the backward finite difference formula and the reproducing kernel method, respectively. And very recently, Gao et al. [9,10] presented some high order difference schemes for both one-dimensional and two-dimensional fractional diffusion equations of distributed order, first by the composite trapezoid formula or the composite Simpson formula to discretize the distributed integral and then by the weighted and shifted Grünwald formula to treat the time-fractional derivatives in the resultant multi-term fractional diffusion equation; among which the alternating direction and the Richardson extrapolation techniques are also involved.

One can observe that the above mentioned numerical methods are all constructed on the basis of the classical numerical quadrature formulae, which have no more than accuracy of order 4 in the distributed-order variable. In contrast, our attention in this paper will be paid on a new class of stable quadrature rules which can achieve accuracy of order arbitrary in theory, and then apply the new quadrature rule to the fourth-order time fractional sub-diffusion equation of the distributed order which is more complex than the time distributed-order sub-diffusion and diffusion-wave equations.

The rest of the paper is organized as follows. In Section 2, we shall review the derivation of the new quadrature rule and introduce some lemmas needed in what follows. In Section 3, a class of new compact difference schemes for the problem (1.1)–(1.4) is derived. In Section 4, the stability and convergence of the suggested scheme are proved by utilizing discrete energy method. Finally, a numerical example is given to illustrate the effectiveness and high accuracy of the suggested scheme.

2. New quadrature rule and some lemmas

In this paper we shall use a new quadrature rule based on boundary value method to approximate the integral term in the distributed-order derivative (1.5). To describe this idea, we consider the quadrature problem

$$I'(\gamma) = \phi(\gamma), \ \gamma \in [0, \gamma_s], \quad I(0) = \phi_0, \tag{2.1}$$

where ϕ is given smooth function. The solution of the problem (2.1) at γ_n is

$$I(\gamma_n) = \phi_0 + \int_0^{\gamma_n} \phi(\gamma)\,d\gamma, \quad \gamma_n \in [0, \gamma_s]. \tag{2.2}$$