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A projection method with regularization for Cauchy problem of the time-harmonic Maxwell equations ☆

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ABSTRACT

We develop a projection method with regularization for reconstructing the radiation electromagnetic field in the exterior of a bounded domain from the knowledge of Cauchy data. The method is divided into two parts. We first solve the complete tangential component of the electrical field on the boundary of that domain from Cauchy data. The radiation electromagnetic field is then recovered from the complete tangential component of the electrical field. For the first part, we transform the Cauchy problem into a compact operator equation by means of the electric-to-magnetic Calderón operator and propose a projection method with regularization to solve that compact operator equation. Meanwhile, we analyze the asymptotic behavior of the singular values of the corresponding compact operator. For the second part, we expend the radiation electromagnetic field to the vector spherical harmonics. Numerical examples are finally presented to demonstrate the computational efficiency of the proposed method.

1. Introduction

We consider in this paper Cauchy problem of the time-harmonic Maxwell Equations. Maxwell equations consist of two pairs of coupled partial differential equations relating six fields, two of which model sources of electromagnetism. This system can be reduced to its time-harmonic form by assuming the propagation of the electromagnetic wave at a single frequency. It arises naturally in many physical applications, such as non-destructive testing of objects by microwave interrogation [20], microwave medical imaging [11,12], design of efficient radiators [3,27], source localization [4] and mine detection [5]. For more details on physical background of electromagnetic wave, we refer the reader to [16,23].

Several prominent methods have been developed for the computational electromagnetics in the past century, such as integral equation methods [8–10,13] and the finite element methods [18,19,22]. Among these methods, the surface tangential components of the electromagnetic field are considered as the input data for computing the electromagnetic field around an object. More recently, some study in the reconstruction of the surface tangential components of the electromagnetic field from near-field measurements has been considered in the works [2,28]. The solution of these methods is not unique unless the input data is provided over a surface enclosing all the sources of the electromagnetic field. As a result, the efficiency of

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these methods weakens when the input data is only available over a portion of a closed surface. However, in many engineering applications the measurements over a closed surface are often infeasible or impossible, in particular for the far-field measurements. We therefore must consider the problem of the analytic continuation of the solution of Maxwell equations in a domain from its surface tangential components on a part of the boundary of this domain, *i.e.* a Cauchy problem of Maxwell equations.

The Cauchy problem for the time-harmonic Maxwell equations is ill-posed. As that for Helmholtz equation, the problem has unique solution but is unstable (see [1,6,15]). The presence of noise in the measurements will be amplified in the solution and in most cases the solution will be useless. For that reason, there is a considerable interest in establishing reliable and fast numerical algorithms for the Cauchy problem of Maxwell equations. A recent method on approximating the solution to this problem may be found in [25,26] and the reference therein.

We consider in this paper the reconstruction of the radiation solution in the exterior of a bounded domain of the time-harmonic Maxwell equations from the knowledge of Cauchy data, which is the surface tangential components of the electromagnetic field on a part of the boundary of the aforementioned domain. Our main idea to deal with this Cauchy problem is divided into two parts. We first recover the complete date of the surface tangential component of the electrical field from Cauchy data. We shall reduce the Cauchy problem to a compact operator equation using the electric-to-magnetic Calderón operator and solve the corresponding compact operator equation by a projection method with regularization strategy. We then find the radiation solution from the Dirichlet boundary condition (the tangential components of the electrical field on the boundary).

This paper is organized in six sections. In Section 2, we introduce the Cauchy problem of Maxwell equations and formulate this problem into a compact operator equation. We discuss in Section 3 the properties of the corresponding compact operator and analyze the asymptotic behavior of the singular values of that operator. A projection method with regularization is proposed in Section 4 for solving the corresponding compact operator equation. In Section 5, we present three numerical examples to confirm the efficiency of the proposed method. The final section summarizes the results of this paper and describes our future works.

2. Formulation of the problems

Cauchy problem of time-harmonic Maxwell equations is presented in this section. By introducing the classical electric-to-magnetic Calderón operator, we reduce the problem to an operator equation, that is determining the complete surface tangential component of the electrical field from Cauchy data.

We begin with describing the Cauchy problem of Maxwell equations. Let B_R be a ball with its center at the origin and of radius $R > 0$. We assume that all the sources of the electromagnetic field are contained in B_R. We consider the time-harmonic electromagnetic wave with frequency $\omega > 0$ propagation in a homogeneous as well as an isotropic medium with space independent electric permittivity $\varepsilon > 0$, magnetic permeability $\mu > 0$ and electric conductivity $\sigma \geq 0$. Let the wave number k be defined by $k^2 = (\varepsilon + \mathrm{i}\sigma/\omega)\mu\omega^2$, where $\mathrm{i} := \sqrt{-1}$ is the imaginary unit. The complex valued space dependent parts $\boldsymbol{E} = [E_1, E_2, E_3]^\top$ of the electric field and $\boldsymbol{H} = [H_1, H_2, H_3]^\top$ of the magnetic field satisfy Maxwell equations

$$\begin{cases} \mathrm{curl}\boldsymbol{E} - \mathrm{i}k\boldsymbol{H} = 0, & \\ \mathrm{curl}\boldsymbol{H} + \mathrm{i}k\boldsymbol{E} = 0, & \text{in } \mathbb{R}^3 \setminus \overline{B_R}, \\ \mathrm{div}\boldsymbol{E} = 0, \text{ and } \mathrm{div}\boldsymbol{H} = 0, & \end{cases} \tag{2.1}$$

with one of the Silver–Müller radiation conditions

$$\lim_{r\to\infty} r(\boldsymbol{H} \times \hat{\boldsymbol{x}} - \boldsymbol{E}) = 0, \text{ or } \lim_{r\to\infty} r(\boldsymbol{E} \times \hat{\boldsymbol{x}} - \boldsymbol{H}) = 0, \tag{2.2}$$

where $\boldsymbol{x} := [x_1, x_2, x_3]^\top \in \mathbb{R}^3$, $r := |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} := \boldsymbol{x}/r$. In this paper, we shall simply call $\boldsymbol{E} = [E_1, E_2, E_3]^\top$ and $\boldsymbol{H} = [H_1, H_2, H_3]^\top$ the electric field and magnetic field, respectively. The Cauchy data is the tangential components of the electrical field and magnetic field on $\Gamma \subsetneq \partial B_R$ given by

$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{f}, \ \boldsymbol{\nu} \times \boldsymbol{H} = \boldsymbol{g}, \text{ on } \Gamma, \tag{2.3}$$

where $\boldsymbol{\nu}$ is the unit outward normal of the boundary ∂B_R, the vector functions $\boldsymbol{f} = [f_1, f_2, f_3]^\top$ and $\boldsymbol{g} = [g_1, g_2, g_3]^\top$ belong to the space of the surface tangential vector field. We shall consider the reconstruction of the electromagnetic field in $\mathbb{R}^3 \setminus B_R$ from Cauchy data, that is to find a radiation solution $\boldsymbol{E}$ and $\boldsymbol{H}$ of the system (2.1)–(2.2) from the Cauchy condition (2.3). We assume that the parameters ε, μ and σ are constants and the wave number k is a positive constant in the following parts of this paper.

We next present some notations and functional spaces to formulate an operator equation. For $n \in \mathbb{N} := \{1, 2, \ldots\}$, let $\mathbb{Z}_n := \{1, 2, \ldots, n\}$ and $\mathbb{Z}_n^c := \mathbb{N} \setminus \mathbb{Z}_n$. The dot product of two vectors $\boldsymbol{a} = [a_1, a_2, a_3]^\top \in \mathbb{C}^3$ and $\boldsymbol{b} = [b_1, b_2, b_3]^\top \in \mathbb{C}^3$ is defined by $\boldsymbol{a} \cdot \boldsymbol{b} := \sum_{j\in\mathbb{Z}_3} a_j \overline{b_j}$. For a continuously differentiable function φ defined on the unit sphere $\mathbb{S}$, we use $\mathrm{Grad}\varphi$ to